ADDENDUM TO
Curriculum Vitae of David W. K. Yeung

Mathematical Formulae Developed

One cannot escape the feeling that these mathematical formulas have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers... ~Heinrich Hertz (1847-1894)

Part A: Control Theory

Theorem A1. (Random-horizon Bellman Equation)

A set of strategies $\{u_k = \psi_k(x)$, for $k \in \mathrm{T}\}$ provides an optimal solution to the control Problem A1 (see below) if there exist functions $V(k,x)$, for $k \in \mathrm{T}$, such that the following recursive relations are satisfied:

$$V(T+1,x) = q_{T+1}(x),$$

$$V(T,x) = \max_{u_T} \left\{ g_T(x,u_T) + V[T+1, f_T(x,u_T)] \right\},$$

$$V(\tau,x) = \max_{u_\tau} \left\{ g_\tau(x,u_\tau) + \frac{\theta_\tau}{\sum_{\zeta=\tau}^{T} \theta_\zeta} q_{\tau+1}[f_\tau(x,u_\tau)] + \frac{\sum_{\zeta=\tau+1}^{T} \theta_\zeta}{\sum_{\zeta=\tau}^{T} \theta_\zeta} V[\tau+1, f_\tau(x,u_\tau)] \right\}, \text{ for } \tau \in \{1,2,\cdots,T-1\}. \quad \blacksquare$$

***Reference**: D.W.K. Yeung and L. A. Petrosyan: Subgame Consistent Cooperative Solution of Dynamic Games with Random Horizon. Journal of Optimization Theory and Applications, Vol. 150, pp78-97, 2011.*

Theorem A2 (Random-horizon Stochastic Bellman Equation under Uncertain Future Payoff Structures)

A set of strategies $\{u_k^{(\sigma_k)*} = \phi_k^{(\sigma_k)*}(x)$, for $\sigma_k \in \{1,2,\cdots,\eta_k\}$ and $k \in \{1,2,\cdots,T\}\}$ provides an optimal solution to the stochastic control Problem A2 if there exist functions $V^{(\sigma_k)}(k,x)$, for $k \in \{1,2,\cdots,T\}$, such that the following recursive relations are satisfied:

$$V^{(\sigma_{T+1})}(T+1,x) = q_{T+1}(x),$$

$$V^{(\sigma_T)}(T,x) = \max_{u_T} E_{\vartheta_T}\left\{ g_T(x,u_T;\theta_T^{\sigma_T}) + V^{(\sigma_{T+1})}[T+1, f_T(x,u_T)+\vartheta_T] \right\}$$

$$= E_{\vartheta_T}\left\{ g_T(x,\phi_T^{(\sigma_T)*}(x);\theta_T^{\sigma_T}) + V^{(\sigma_{T+1})}[T+1, f_T(x,\phi_T^{(\sigma_T)*}(x))+\vartheta_T] \right\},$$

$$V^{(\sigma_\tau)}(\tau,x) = \max_{u_\tau} E_{\vartheta_\tau}\left\{ g_\tau(x,u_\tau;\theta_\tau^{\sigma_\tau}) + \frac{\varpi_\tau}{\sum_{\zeta=\tau}^{T}\varpi_\zeta} q_{\tau+1}[f_\tau(x,u_\tau)+\vartheta_\tau] \right.$$

$$\left. + \frac{\sum_{\zeta=\tau+1}^{T}\varpi_\zeta}{\sum_{\zeta=\tau}^{T}\varpi_\zeta} \sum_{\sigma_{\tau+1}=1}^{\eta_{\tau+1}} \lambda_{\tau+1}^{\sigma_{\tau+1}} V^{(\sigma_{\tau+1})}[\tau+1, f_\tau(x,u_\tau)+\vartheta_\tau] \right\}$$

$$= E_{\vartheta_\tau}\left\{ g_\tau(x,\phi_\tau^{(\sigma_\tau)*}(x);\theta_\tau^{\sigma_\tau}) + \frac{\varpi_\tau}{\sum_{\zeta=\tau}^{T}\varpi_\zeta} q_{\tau+1}[f_\tau(x,\phi_\tau^{(\sigma_\tau)*}(x))+\vartheta_\tau] \right.$$

$$\left. + \frac{\sum_{\zeta=\tau+1}^{T}\varpi_\zeta}{\sum_{\zeta=\tau}^{T}\varpi_\zeta} \sum_{\sigma_{\tau+1}=1}^{\eta_{\tau+1}} \lambda_{\tau+1}^{\sigma_{\tau+1}} V^{(\sigma_{\tau+1})}[\tau+1, f_\tau(x,\phi_\tau^{(\sigma_\tau)*}(x))+\vartheta_\tau] \right\},$$

for $\tau \in \{1,2,\cdots,T-1\}$. ■

Reference*: *D.W.K. Yeung and L.A. Petrosyan: Subgame Consistent Cooperative Solutions For Randomly Furcating Stochastic Dynamic Games With Uncertain Horizon. International Game Theory Review, Vol. 16, 2014, pp.1440012.01-1440012.29.

Theorem A.3. Dynamic Optimization Technique for Durable Controls

Let $W(k,x;u_{k-})$ be the maximal value of the payoff of the problem of maximizing

$$\sum_{k=1}^{T} g_k(x_k,u_k;u_{k-}) + q_{T+1}(x_{T+1};u_{(T+1)-})$$

subject to the dynamics $x_{k+1} = f_k(x_k,u_k;u_{k-})$, $\quad x_1 = x_1^0$,

where u_{k-} is the set of controls which are executed before stage k but still in effect in stage k.

The function $W(k,x;u_{k-})$ satisfies the following system of recursive equations:

$$W(T+1,x;u_{(T+1)-}) = q_{T+1}(x_{T+1};u_{(T+1)-}),$$

$$W(k,x;u_{k-}) = \max_{u_k}\left\{ g_k(x,u_k;u_{k-}) + W[k+1, f_k(x,u_k;u_{k-});u_{(k+1)-}] \right\},$$

for $k \in \{1,2,\cdots,T\}$. ■

***References*: *D.W.K. Yeung, L.A. Petrosyan (2020): Cooperative Dynamic Games with Durable Controls: Theory and Application, Dynamic Games and Applications, Doi:10.1007/s13235-019-00336-w.*
9(2), 550-567, 2019, https://doi.org/10.1007/s13235-018-0266-6.

D.W.K. Yeung, L.A. Petrosyan (2019): Cooperative Dynamic Games with Control Lags, Dynamic Games and Applications, 9(2), 550-567, https://doi.org/10.1007/s13235-018-0266-6.

Part B: Game Theory

Theorem B1. (Random-horizon (Hamilton-Jacobi-Bellman) HJB Equations)

A set of strategies $\{\phi_k^i(x)$, for $k \in T$ and $i \in N\}$ provides a feedback Nash equilibrium solution to the game Problem B1 (see below) if there exist functions $V^i(k,x)$, for $k \in T$ and $i \in N$, such that the following recursive relations are satisfied:

$$V^i(T,x) = \max_{u_T^i} \Big\{ g_T^i[x,\phi_T^1(x),\phi_T^2(x),\cdots,\phi_T^{i-1}(x),u_T^i,\phi_T^{i+1}(x),\cdots,\phi_T^n(x)]$$

$$+ q_{T+1}^i [f_k(x,\phi_k^1(x),\phi_k^2(x),\cdots,\phi_k^{i-1}(x),u_k^i,\phi_k^{i+1}(x),\cdots,\phi_k^n(x))] \Big\} ,$$

$$V^i(\tau,x) = \max_{u_\tau^i} \Big\{ g_\tau^i[x,\phi_\tau^1(x),\phi_\tau^2(x),\cdots,\phi_\tau^{i-1}(x),u_\tau^i,\phi_\tau^{i+1}(x),\cdots,\phi_\tau^n(x)]$$

$$+ \frac{\theta_\tau}{\sum_{\zeta=\tau}^{T}\theta_\zeta} q_{\tau+1}^i [f_k(x,\phi_k^1(x),\phi_k^2(x),\cdots,\phi_k^{i-1}(x),u_k^i,\phi_k^{i+1}(x),\cdots,\phi_k^n(x))]$$

$$+ \frac{\sum_{\zeta=\tau+1}^{T}\theta_\zeta}{\sum_{\zeta=\tau}^{T}\theta_\zeta} V^i[\tau+1, f_k(x,\phi_k^1(x),\phi_k^2(x),\cdots,\phi_k^{i-1}(x),u_k^i,\phi_k^{i+1}(x),\cdots,\phi_k^n(x))] . \Big\}, \quad (3.1)$$

for $\tau \in \{1,2,\cdots,T-1\}$. ■

***Reference*: *D.W.K. Yeung and L. A. Petrosyan: Subgame Consistent Cooperative Solution of Dynamic Games with Random Horizon. Journal of Optimization Theory and Applications, Vol. 150, pp78-97, 2011.*

Theorem B2. (Random-horizon Stochastic HJB Equations under

Uncertain Future Payoff Structures)

A set of strategies $\{u_\tau^{i*} = \phi_\tau^{(\sigma_\tau)i*}(x)$, for $\sigma_\tau \in \{1,2,\cdots,\eta_\tau\}$, $\tau \in \{1,2,\cdots,T\}$ and $i \in N\}$ constitutes a Nash equilibrium solution to the game Problem B2 (see below) if there exist functions $V^{(\sigma_\tau)i}(\tau,x)$, for $\sigma_\tau \in \{1,2,\cdots,\eta_\tau\}$, $\tau \in \{1,2,\cdots,T\}$ and $i \in N$, such that the following recursive relations are satisfied:

$$V^{(\sigma_{T+1})i}(T+1,x) = q_{T+1}^i(x),$$

$$V^{(\sigma_T)i}(T,x) = \max_{u_T^i} E_{\vartheta_T}\left\{ g_T^i[x,\phi_T^{(\sigma_T)1*}(x),\phi_T^{(\sigma_T)2*}(x),\cdots,\phi_T^{(\sigma_T)i-1*}(x),u_T^{(\sigma_T)i},\phi_T^{(\sigma_T)i+1*}(x),\cdots \right.$$
$$\left. \cdots,\phi_T^{(\sigma_T)n*}(x)\,;\theta_T^{\sigma_T}] + V^{(\sigma_{T+1})i}[T+1, f_T\,(x,\underline{\phi}_T^{(\sigma_T)*\neq i}) + \vartheta_T] \right\},$$

$$V^{(\sigma_\tau)i}(\tau,x) = \max_{u_\tau^i} E_{\vartheta_\tau}\left\{ g_\tau^i[x,\phi_\tau^{(\sigma_\tau)1*}(x),\phi_\tau^{(\sigma_\tau)2*}(x),\cdots,\phi_\tau^{(\sigma_\tau)i-1*}(x),u_\tau^{(\sigma_\tau)i},\phi_\tau^{(\sigma_\tau)i+1*}(x),\cdots \right.$$
$$\cdots,\phi_\tau^{(\sigma_\tau)n*}(x)\,;\theta_\tau^{\sigma_\tau}] + \frac{\varpi_\tau}{\sum_{\zeta=\tau}^{T}\varpi_\zeta} q_{\tau+1}[\,f_\tau\,(x,\underline{\phi}_\tau^{(\sigma_\tau)*\neq i}(x)) + \vartheta_\tau\,]$$
$$\left. + \frac{\sum_{\zeta=\tau+1}^{T}\varpi_\zeta}{\sum_{\zeta=\tau}^{T}\varpi_\zeta} \sum_{\sigma_{\tau+1}=1}^{\eta_{\tau+1}} \lambda_{\tau+1}^{\sigma_{\tau+1}}\, V^{(\sigma_{\tau+1})}[\tau+1, f_\tau\,(x,\underline{\phi}_\tau^{(\sigma_\tau)*\neq i}(x)) + \vartheta_\tau\,] \right\},\ \tau \in \{1,2,\cdots,T-1\};$$

for $\sigma_t \in \{1,2,\cdots,\eta_t\}$, $t \in \{1,2,\cdots,T\}$ and $i \in N$;

where $\underline{\phi}_t^{(\sigma_t)*\neq i}(x)$

$$= [\phi_t^{(\sigma_t)1*}(x),\phi_t^{(\sigma_t)2*}(x),\cdots,\phi_t^{(\sigma_t)i-1*}(x),u_t^{(\sigma_t)i},\phi_t^{(\sigma_t)i+1*}(x),\cdots,\phi_t^{(\sigma_t)n*}(x)];$$

for $t \in \{1,2,\cdots,T\}$. ■

Reference: *D.W.K. Yeung and L.A. Petrosyan: Subgame Consistent Cooperative Solutions For Randomly Furcating Stochastic Dynamic Games With Uncertain Horizon. International Game Theory Review, Vol. 16, 2014, pp.1440012.01-1440012.29.*

Theorem B4. (Nontransferable Individual Payoff in Continuous-time Stochastic Dynamic Cooperation)

If there exists a set of controls $\left\{u_i^{(\alpha)}(t) = \psi_i^{(\alpha)}(t,x),\ \text{for } i \in N\right\}$ and value functions $W^{(\alpha)}(t,x) : [t_0,T]\times R^n \to R$ which provide an optimal solution to the stochastic control Problem B4 (see below), then the individual player's payoff $W^{(\alpha)i}(t,x) : [t_0,T]\times R^n \to R$ for $i \in N$ satisfy the following partial differential equations:

$$-W_t^{(\alpha)}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}(t,x)W_{x^h x^\zeta}^{(\alpha)}(t,x) =$$

$$\max_{u_1,u_2}\Bigg\{ \left(\sum_{j=1}^{n}\alpha^j g^j(t,x,u_1,u_2,\cdots,u_n)\right)\exp\left[-\int_{t_0}^{t} r(y)dy\right]$$

$$+W_x^{(\alpha)}(t,x)\ f(t,x,u^1,u^2,\cdots,u^n)\Bigg\},$$

$$W^{(\alpha)}(T,x)=\exp[-r(T-t_0)]\sum_{j=1}^{n}\alpha^j q^j(x),$$

$$-W_t^{(\alpha)i}(t,x)-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}(t,x)W_{x^h x^\zeta}^{(\alpha)i}(t,x)=$$

$$g^i[t,x,\psi_1^{(\alpha)}(t,x),\psi_2^{(\alpha)}(t,x),\cdots,\psi_n^{(\alpha)}(t,x)]\exp\left[-\int_{t_0}^{t} r(y)dy\right]$$

$$+W_x^{(\alpha)i}(t,x)\ f[t,x,\psi_1^{(\alpha)}(t,x),\psi_2^{(\alpha)}(t,x),\cdots,\psi_n^{(\alpha)}(t,x)] \text{ and}$$

$$W^{(\alpha)i}(T,x)=\exp\left[-\int_{t_0}^{T} r(y)dy\right]q^i(x), \qquad \text{for } i\in N. \qquad \blacksquare$$

References: *D.W.K. Yeung: Nontransferable Individual Payoff Functions under Stochastic Dynamic Cooperation, International Game Theory Review, Vol. 6, 2004, pp. 281-289.*

D.W.K. Yeung: Nontransferable Individual Payoffs in Cooperative Stochastic Dynamic Games, International Journal of Algebra, Vol. 7, 2013, pp. 597-606.

Theorem B5. (Subgame-consistent Payoff Distribution Procedure (PDP) for Discrete-time Stochastic Dynamic Cooperation)

Consider the cooperative stochastic dynamic game Problem B5 (see below) in which the players agree to maximize their joint expected payoff and share the cooperative gain according to the imputation $\xi^i(k,x_k^*)$ for player $i\in N$ in stage $k\in\kappa$ along the cooperative trajectory $\{x_k^*\}_{k=1}^{T}$. A Payoff Distribution Procedure (PDP) with a payment equaling

$$B_k^i(x_k^*)=(1+r)^{k-1}\Bigg\{\ \xi^i(k,x_k^*)$$

$$-E_{\theta_k}\left(\ \xi^i[k+1,f_k\ (x_k^*,\psi_k^1(x_k^*),\psi_k^2(x_k^*),\cdots,\psi_k^n(x_k^*))+\theta_k\,]\ \right)\Bigg\},$$

for $i\in N$,

given to player i at stage $k\in\kappa$, if $x_k^*\in X_k^*$ would lead to the realization of the imputation { $\xi^i(k,x_k^*)$, for $i\in N$ and $k\in\kappa$ };

where

$\{\psi_k^i(x)$, for $k\in\kappa$ and $i\in N\}$ is a set of strategies that provides an optimal solution to the Problem B5 yielding functions $W(k,x)$, for $k\in K$, such that the following recursive relations are satisfied:

$$W(k,x)=\max_{u_k^1,u_k^2,\cdots,u_k^n} E_{\theta_k}\Bigg\{\ \sum_{j=1}^{n}\ g_k^j[x,u_k^1,u_k^2,\cdots,u_k^n]\left(\frac{1}{1+r}\right)^{k-1}$$

$$+W[k+1, f_k(x,u_k^1,u_k^2,\cdots,u_k^n)+\theta_k] \quad \Big\},$$

$$W(T+1,x)=\sum_{j=1}^{n} q_{T+1}^{j}(x)\left(\frac{1}{1+r}\right)^{T}. \qquad ■$$

References: *D.W.K. Yeung and L. A. Petrosyan: Subgame Consistent Solutions for Cooperative Stochastic Dynamic Games. Journal of Optimization Theory and Applications, Vol. 145, 2010, pp. 579-596.*

Problem B5: ***Corresponding Problem of Theorem B5.***

Consider the general $T-$ stage $n-$ person discrete-time cooperative stochastic dynamic game with initial state x^0. The state space of the game is $X \in R^m$ and the state dynamics of the game is characterized by the stochastic difference equation:

$$x_{k+1}=f_k(x_k,u_k^1,u_k^2,\cdots,u_k^n)+\theta_k,$$

for $k \in \{1,2,\cdots,T\} \equiv \kappa$ and $x_1 = x^0$,

where $u_k^i \in R^{m_i}$ is the control vector of player i at stage k, $x_k \in X$ is the state, and θ_k is a set of statistically independent random variables.

The objective of player i is

$$E_{\theta_1,\theta_2,\cdots,\theta_T}\left\{ \sum_{\zeta=1}^{T} g_\zeta^i[x_\zeta,u_\zeta^1,u_\zeta^2,\cdots,u_\zeta^n]\left(\frac{1}{1+r}\right)^{\zeta-1} + q_{T+1}^i(x_{T+1})\left(\frac{1}{1+r}\right)^{T} \right\},$$

for $i \in \{1,2,\cdots,n\} \equiv N$,

where r is the discount rate and $E_{\theta_1,\theta_2,\cdots,\theta_T}$ is the expectation operation with respect to the statistics of $\theta_1,\theta_2,\cdots,\theta_T$.

The players agree to maximize their joint expected payoff and share the cooperative gain according to the imputation $\xi^i(k,x_k^*)$ for player $i \in N$ in stage $k \in \kappa$ along the cooperative trajectory $\{x_k^*\}_{k=1}^{T}$. Examples of the imputation $\xi^i(k,x_k^*)$ include:

(i) Sharing the extra gain from cooperation equally, and the imputation to player i becomes:

$$\xi^i(k,x_k^*)= V^i(k,x_k^*)+\frac{1}{n}\left[W(k,x_k^*)-\sum_{j=1}^{n}V^j(k,x_k^*)\right], \quad \text{for } i \in N \text{ and } k \in \kappa,$$

where $V^i(k,x_k^*)$ is the expected noncooperative payoff of player i and $W(k,x_k^*)$ is the expected total cooperative payoff.

(ii) Share the total cooperative proportional to the players' noncooperative payoffs, and the imputation to player i becomes:

$$\xi^i(k,x_k^*)= \frac{V^i(k,x_k^*)}{\sum_{j=1}^{n}V^j(k,x_k^*)}W(k,x_k^*), \qquad \text{for } i \in N \text{ and } k \in \kappa.$$

Theorem B6. (Subgame-consistent PDP for Continuous-time Stochastic Dynamic Cooperation)

Consider the cooperative stochastic differential game Problem B6 (see below) in which the players agree to maximize their joint expected payoff and share the cooperative gain according to the imputation $\xi^{(s)i}(s, x_s^*)$ in current value at time s for player $i \in N$ in time $s \in [t_0, T]$ along the cooperative trajectory $\{x_s^*\}_{s=t_0}^T$. A Payoff Distribution Procedure (PDP) with a payment equaling

$$B_i\,(s, x_s^*) \;=\; -\left[\xi_t^{(s)i}(t, x_t^*)\Big|_{t=s}\right]$$

$$-\left[\xi_{x_t^*}^{(s)i}(t, x_t^*)\Big|_{t=s}\right] f[s, x_s^*, \psi_1^*(s, x_s^*), \psi_2^*(s, x_s^*), \cdots, \psi_n^*(s, x_s^*)]$$

$$-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}(s, x_s^*)\left[\xi_{x_t^h x_t^\zeta}^{(s)i}(t, x_t^*)\Big|_{t=s}\right], \qquad \text{for } i \in N \text{ and } x_s^* \in X_s^*,$$

given to player i at time $s \in [t_0, T]$ would lead to the realization of the imputation $\{\xi^{(s)i}(s, x_s^*)$, for $i \in N$ and $s \in [t_0, T]\}$;

where

$\{\psi_i^*(s, x)$, for $i \in N$ and $s \in [t_0, T]$ is a set of strategies that provides an optimal solution to the Problem B6 yielding functions continuously twice differentiable functions $W(t, x)$:[t_0,T] $\times R^m \to R$, which satisfy the following partial differential equation:

$$-W_t\,(t, x) - \frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}(t, x) W_{x^h x^\zeta}(t, x) =$$

$$\max_{u_1, u_2, \cdots, u_n}\left\{\sum_{j=1}^{n} g^j[t, x, u_1, u_2, \cdots, u_n]\exp\left[-\int_{t_0}^{t} r(y)dy\right]\right.$$

$$\left. + W_x\,(t, x) f[t, x, u_1, u_2, \cdots, u_n]\right\}, \text{ and}$$

$$W(T, x) = \sum_{j=1}^{n} q^j(x)\exp\left[-\int_{t_0}^{T} r(y)dy\right]. \qquad \blacksquare$$

References: *D.W.K. Yeung and L. Petrosyan: Subgame Consistent Cooperative Solution in Stochastic Differential Games, Journal of Optimization Theory and Applications, Vol. 120, 2004, pp.651-666.*

D.W.K. Yeung and L. A. Petrosyan: Subgame Consistent Economic Optimization: An Advanced Cooperative Dynamic Game Analysis, Boston: Birkhäuser. ISBN 978-0-8176-8261-3, 395pp, 2012.

Problem B6: ***Corresponding Problem of Theorem B6.***

Consider the n-person cooperative stochastic differential game in which player i seeks to maximize its expected payoffs:

$$E_{t_0}\left\{\int_{t_0}^{T} g^i[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)]\exp\left[-\int_{t_0}^{s} r(y)dy\right]ds\right.$$

$$+\exp\left[-\int_{t_0}^{T} r(y)dy\right] q^i(x(T)) \quad \Bigg\},\qquad \text{for } i\in N, \qquad \text{with}$$

$E_{t_0}\{\cdot\}$ denoting the expectation operation taken at time t_0, and the dynamics of the state is

$$dx(s) = f[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)]ds + \sigma[s, x(s)]dz(s), \quad x(t_0) = x_0,$$

where $\sigma[s, x(s)]$ is a $m \times \Theta$ matrix and $z(s)$ is a Θ-dimensional Wiener process and the initial state x_0 is given. Let $\Omega[s, x(s)] = \sigma[s, x(s)]\ \sigma[s, x(s)]'$ denote the covariance matrix with its element in row h and column ζ denoted by $\Omega^{h\zeta}[s, x(s)]$. Moreover, $E[dz_\varpi] = 0$ and $E[dz_\varpi dt] = 0$ and $E[(dz_\varpi)^2] = dt$, for $\varpi \in [1,2,\cdots,\Theta]$; and $E[dz_\varpi dz_\omega] = 0$, for $\varpi \in [1,2,\cdots,\Theta]$, $\omega \in [1,2,\cdots,\Theta]$ and $\varpi \neq \omega$.

The players agree to maximize their joint expected payoff and share the cooperative gain according to the imputation $\xi^{(s)i}(s, x_s^*)$ in current value at time s for player $i \in N$ in time $s \in [t_0, T]$ along the cooperative trajectory $\{x_s^*\}_{s=t_0}^{T}$.

Theorem B7. (Subgame-consistent PDP for Random-horizon Dynamic Cooperation)

Consider the random-horizon cooperative dynamic game Problem 7 (see below) in which the players agree to maximize their joint payoff and share the cooperative gain according to the imputation $\xi^i(\tau, x_\tau^*)$ for player $i \in N$ in stage $\tau \in \kappa$ along the cooperative trajectory $\{x_\tau^*\}_{\tau=1}^{T}$. A Payoff Distribution Procedure (PDP) with a payment equaling

$$B_\tau^i(x_\tau^*) = \xi^i(\tau, x_\tau^*) - \frac{\sum_{\zeta=\tau+1}^{T}\theta_\zeta}{\sum_{\zeta=\tau}^{T}\theta_\zeta}\ \xi^i(\tau+1,\ f_\tau[x_\tau, \psi_\tau^1(x_\tau), \psi_\tau^2(x_\tau), \cdots, \psi_\tau^n(x_\tau)])$$

$$-\frac{\theta_\tau}{\sum_{\zeta=\tau}^{T}\theta_\zeta} q_{\tau+1}^i(f_\tau[x_\tau, \psi_\tau^1(x_\tau), \psi_\tau^2(x_\tau), \cdots, \psi_\tau^n(x_\tau)]), \quad \text{for } i \in N,$$

given to player i at stage $\tau \in \kappa$ would lead to the realization of the imputation $\xi^i(\tau, x_\tau^*)$ for player $i \in N$ in stage $\tau \in \kappa$;

where

$\{\psi_\tau^i(x)$, for $\tau \in \kappa$ and $i \in N\}$ is a set of strategies that provides a group optimal solution to the Problem 9 yielding functions $W(k, x)$, for $\tau \in \kappa$, such that the following recursive relations are satisfied:

$$W(T+1, x) = \sum_{j=1}^{n} q_{T+1}^j(x),$$

$$W(T, x) = \max_{u_T^1, u_T^2, \cdots, u_T^n} \left\{ \sum_{j=1}^{n} g_T^j[x, u_T^1, u_T^2, \cdots, u_T^n] + q_{T+1}[f_T(x, u_T, u_T^1, u_T^2, \cdots, u_T^n)] \right\},$$

$$W(\tau,x)=\max_{u_\tau^1,u_\tau^2,\cdots,u_\tau^n}\left\{\sum_{j=1}^{n}\left[g_\tau^j[x,u_\tau^1,u_\tau^2,\cdots,u_\tau^n]+\frac{\theta_\tau}{\sum_{\zeta=\tau}^{T}\theta_\zeta}q_{\tau+1}^j[f_\tau(x,u_\tau^1,u_\tau^2,\cdots,u_\tau^n)]\right]\right.$$

$$\left.+\frac{\sum_{\zeta=\tau+1}^{T}\theta_\zeta}{\sum_{\zeta=\tau}^{T}\theta_\zeta}W[\tau+1,f_\tau(x,u_\tau^1,u_\tau^2,\cdots,u_\tau^n)]\right\},\text{ for }\tau\in\{1,2,\cdots,T-1\}.\quad\blacksquare$$

References: *D.W.K. Yeung and L. A. Petrosyan: Subgame Consistent Cooperative Solution of Dynamic Games with Random Horizon. Journal of Optimization Theory and Applications, Vol. 150, pp78-97, 2011.*

Problem B7: ***Corresponding Problem of Theorem B7.***

Consider the $n-$person cooperative dynamic game with $\hat{T}$ stages where $\hat{T}$ is a random variable with range $\{1,2,\cdots,T\}$ and corresponding probabilities $\{\theta_1,\theta_2,\cdots,\theta_T\}$. Conditional upon the reaching of stage τ, the probability of the game would last up to stages $\tau,\tau+1,\cdots,T$ becomes respectively

$$\frac{\theta_\tau}{\sum_{\zeta=\tau}^{T}\theta_\zeta},\frac{\theta_{\tau+1}}{\sum_{\zeta=\tau}^{T}\theta_\zeta},\cdots,\frac{\theta_T}{\sum_{\zeta=\tau}^{T}\theta_\zeta}.$$

The payoff of player i at stage $k\in\{1,2,\cdots,T\}$ is $g_k^i[x_k,u_k^1,u_k^2,\cdots,u_k^n]$. When the game ends after stage $\hat{T}$, player i will receive a terminal payment $q_{\hat{T}+1}^i(x_{\hat{T}+1})$ in stage $\hat{T}+1$.

The state space of the game is $X\in R^m$ and the state dynamics of the game is characterized by the difference equation:

$$x_{k+1}=f_k(x_k,u_k^1,u_k^2,\cdots,u_k^n),$$

for $k\in\{1,2,\cdots,T\}\equiv\kappa$ and $x_1=x^0$,

where $u_k^i\in R^{m_i}$ is the control vector of player i at stage k and $x_k\in X$ is the state.

The objective of player i is

$$E\left\{\sum_{k=1}^{\hat{T}}g_k^i[x_k,u_k^1,u_k^2,\cdots,u_k^n]+q_{\hat{T}+1}^i(x_{\hat{T}+1})\right\}$$

$$=\sum_{\hat{T}=1}^{T}\theta_{\hat{T}}\left\{\sum_{k=1}^{\hat{T}}g_k^i[x_k,u_k^1,u_k^2,\cdots,u_k^n]+q_{\hat{T}+1}^i(x_{\hat{T}+1})\right\},$$

for $i\in\{1,2,\cdots,n\}\equiv N$.

The players agree to maximize their joint expected payoff and share the cooperative gain according to the imputation $\xi^i(k,x_k)$ for player $i\in N$ in stage $k\in\kappa$ along the cooperative trajectory $\{x_k\}_{k=1}^T$.

Theorem B8. (Subgame-consistent PDP for Discrete-time Stochastic Dynamic Cooperation under Uncertainty in Payoff Structures)

Consider the randomly furcating cooperative stochastic dynamic game Problem B8 (see below) in which the players agree to maximize their joint expected payoff and share the cooperative gain according to the imputation $\xi^{(\sigma_k)}(k, x_k^*) = [\xi^{(\sigma_k)1}(k, x_k^*), \xi^{(\sigma_k)2}(k, x_k^*), \cdots, \xi^{(\sigma_k)n}(k, x_k^*)]$ along the cooperative trajectory given that $\theta_k^{\sigma_k}$ has occurred in stage k, for $\sigma_k \in \{1,2,\cdots,\eta_k\}$ and $k \in \{1,2,\cdots,T\}$. A Payoff Distribution Procedure (PDP) with a payment equaling

$$B_k^{(\sigma_k)i}(x_k^*) = \xi^{(\sigma_k)i}(k, x_k^*) - E_{\vartheta_k}\left[\sum_{\sigma_{k+1}=1}^{\eta_{k+1}} \lambda_{k+1}^{\sigma_{k+1}} \left(\xi^{(\sigma_{k+1})i}[k+1, f_k(x_k^*, \psi_k^{(\sigma_k)*}(x_k^*)) + \vartheta_k] \right) \right],$$

for $i \in N$,

given to player i at stage $k \in \{1,2,\cdots,T\}$, if $\theta_k^{\sigma_k}$ occurs and $x_k^* \in X_k^*$, leads to the realization of the imputation $\xi^{(\sigma_k)}(k, x_k^*)$ for $k \in \{1,2,\cdots,T\}$;

where

where $\psi_t^{(\sigma_t)*}(x) = \{\psi_t^{(\sigma_t)1*}(x), \psi_t^{(\sigma_t)2*}(x), \cdots \cdots, \psi_t^{(\sigma_t)n*}(x)\}$, for $\sigma_t \in \{1,2,\cdots,\eta_t\}$ and $t \in \{1,2,\cdots,T\}$ is a set of strategies that provides a group optimal solution to Problem B8 yielding value functions $W^{(\sigma_t)}(t, x)$, for $\sigma_t \in \{1,2,\cdots,\eta_t\}$ and $t \in \{1,2,\cdots,T\}$, such that the following recursive relations are satisfied:

$$W^{(\sigma_T)}(T+1, x) = \sum_{j=1}^{n} q^j(x),$$

$$W^{(\sigma_T)}(T, x) = \max_{u_T^{(\sigma_T)1}, u_T^{(\sigma_T)2}, \cdots, u_T^{(\sigma_T)n}} E_{\vartheta_T}\left\{ \sum_{j=1}^{n} g_T^j[x, u_T^{(\sigma_T)1}, u_T^{(\sigma_T)2}, \cdots, u_T^{(\sigma_T)n}; \theta_T^{\sigma_T}] + W^{(\sigma_{T+1})}[T+1, f_T(x, u_T^{(\sigma_T)1}, u_T^{(\sigma_T)2}, \cdots, u_T^{(\sigma_T)n}) + \vartheta_T] \right\},$$

$$W^{(\sigma_t)}(t, x) = \max_{u_t^{(\sigma_t)1}, u_t^{(\sigma_t)2}, \cdots, u_t^{(\sigma_t)n}} E_{\vartheta_t}\left\{ \sum_{j=1}^{n} g_t^j[x, u_t^{(\sigma_t)1}, u_t^{(\sigma_t)2}, \cdots, u_t^{(\sigma_t)n}; \theta_t^{\sigma_t}] + \sum_{\sigma_{t+1}=1}^{\eta_{t+1}} \lambda_{t+1}^{\sigma_{t+1}} W^{(\sigma_{t+1})}[t+1, f_t(x, u_t^{(\sigma_t)1}, u_t^{(\sigma_t)2}, \cdots, u_t^{(\sigma_t)n}) + \vartheta_t] \right\},$$

for $\sigma_t \in \{1,2,\cdots,\eta_t\}$ and $t \in \{1,2,\cdots,T-1\}$. ■

Reference: *D.W.K. Yeung and L. A. Petrosyan: Subgame-consistent Cooperative Solutions in Randomly Furcating Stochastic Dynamic Games. Mathematical and Computer Modelling, Vol 57, pp.976–991, 2013.*

Problem B8: ***Corresponding Problem of Theorem B8.***

Consider the $T-$stage $n-$person randomly furcating cooperative stochastic dynamic game with initial state x^0. The state space of the game is $X \in R^m$ and the state dynamics of the game is characterized by the stochastic difference equation:

$$x_{k+1} = f_k(x_k, u_k^1, u_k^2, \cdots, u_k^n) + \vartheta_k,$$

for $k \in \{1,2,\cdots,T\}$ and $x_1 = x^0$, where $u_k^i \in U^i \subset R^{m_i}$ is the control vector of player i at stage k, $x_k \in X$ is the state, and ϑ_k is a sequence of statistically independent random variables.

The payoff of player i at stage k is $g_k^i(x_k, u_k^1, u_k^2, \cdots, u_k^n; \theta_k)$ which is affected by a random variable θ_k. In particular, θ_k for $k \in \{1,2,\cdots,T\}$ are independent discrete random variables with range $\{\theta_k^1, \theta_k^2, \cdots, \theta_k^{\eta_k}\}$ and corresponding probabilities $\{\lambda_k^1, \lambda_k^2, \cdots, \lambda_k^{\eta_k}\}$, where η_k is a positive integer for $k \in \{1,2,\cdots,T\}$. In stage 1, it is known that θ_1 equals θ_1^1 with probability $\lambda_1^1 = 1$.

The objective that player i seeks to maximize is

$$E_{\theta_1,\theta_2,\cdots,\theta_T;\vartheta_1,\vartheta_2,\cdots,\vartheta_T}\left\{ \sum_{k=1}^{T} g_k^i(x_k, u_k^1, u_k^2, \cdots, u_k^n; \theta_k) + q^i(x_{T+1}) \right\},$$

for $i \in \{1,2,\cdots,n\} \equiv N$,

where $E_{\theta_1,\theta_2,\cdots,\theta_T;\vartheta_1,\vartheta_2,\cdots,\vartheta_T}$ is the expectation operation with respect to the random variables $\theta_1, \theta_2, \cdots, \theta_T$ and $\vartheta_1, \vartheta_2, \cdots, \vartheta_T$, and $q^i(x_{T+1})$ is a terminal payment given at stage $T+1$. The payoffs of the players are transferable.

Theorem B9. Subgame-consistent PDP for Continuous-time Stochastic Dynamic Cooperation under Uncertainty in Payoff Structures

Consider the randomly furcating cooperative stochastic dynamic game Problem B.9 (see below) in which the players agree to maximize their joint expected payoff and share the cooperative gain according to the imputation $\xi^{i[\theta_{a_k}^k](k)\tau}(t, x_t^*)$, for $i \in N$, $\tau \in [t_k, t_{k+1}]$, $t \in [\tau, t_{k+1}]$, $k \in \{0,1,2,\cdots,m-1\}$, and $\theta_{a_k}^h \in \{\theta_1, \theta_2, ..., \theta_\eta\}$. A Payoff Distribution Procedure (PDP) with a payment equaling

$$\begin{aligned} B_i^{(\theta_{a_k}^k)k}(\tau) = & -\left[\xi_t^{i[\theta_{a_k}^k](k)\tau}(t, x_t^*)\Big|_{t=\tau}\right] \\ & -\left[\xi_{x_t^*}^{i[\theta_{a_k}^k](k)\tau}(t, x_t^*)\Big|_{t=\tau}\right] f[\tau, x_\tau^*, \psi_1^{(k)\theta_{a_k}^k}(\tau, x_\tau^*), \psi_2^{(k)\theta_{a_k}^k}(\tau, x_\tau^*), \cdots, \psi_n^{(k)\theta_{a_k}^k}(\tau, x_\tau^*)] \\ & -\frac{1}{2}\sum_{h,\zeta=1}^{n} \Omega^{h\zeta}(\tau, x_\tau^*)\left[\xi_{x_t^h x_t^\zeta}^{i[\theta_{a_k}^k](k)\tau}(t, x_t^*)\Big|_{t=\tau}\right], \end{aligned}$$

for $i \in N$ and $k \in \{1,2,\cdots,m\}$,

given to player i at time $\tau \in [t_k, t_{k+1}]$ contingent upon $\theta_{a_k}^k \in \{\theta_1, \theta_2, ..., \theta_\eta\}$ has occurred at time t_k, leads to the realization of the imputation $\xi^{i[\theta_{a_k}^k](k)\tau}(t, x_t^*)$, for $i \in N$, $\tau \in [t_k, t_{k+1}]$, $t \in [\tau, t_{k+1}]$, $k \in \{0,1,2,\cdots,m-1\}$, and $\theta_{a_k}^h \in \{\theta_1, \theta_2, ..., \theta_\eta\}$.

where

$\{u_i^{(m)\theta_{\alpha_m}^m}(t) = \psi_i^{(m)\theta_{\alpha_m}^m}(t,x)$, for $t \in [t_m, T]$; $u_i^{(k)\theta_{\alpha_k}^k}(t) = \psi_i^{(k)\theta_{\alpha_k}^k}(t,x)$, for $t \in [t_k, t_{k+1})$, $k \in \{0,1,2,\cdots,m-1\}$ and $i \in N\}$, contingent upon the events $\theta_{\alpha_m}^m$ and $\theta_{\alpha_k}^k$ is a set of controls that provides a group optimal solution for the game Problem 11 yielding continuously differentiable functions $W^{[\theta_{\alpha_m}^m](m)}(t,x) : [t_m, T] \times R^\kappa \to R$ and

$W^{[\theta_{\alpha_k}^k](k)}(t,x):[t_k, t_{k+1}] \times R^\kappa \to R$ for $k \in \{0,1,2,\cdots,m-1\}$ which satisfy the following partial differential equations:

$$-W_t^{[\theta_{\alpha_m}^m](m)}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{n}\Omega^{h\zeta}(t,x)W_{x^h x^\zeta}^{[\theta_{\alpha_m}^m](m)}(t,x)$$

$$= \max_{u_1^{\theta_{\alpha_m}}, u_2^{\theta_{\alpha_m}}, \cdots, u_n^{\theta_{\alpha_m}}} \left\{ \sum_{j=1}^{n} g^{[j,\theta_{\alpha_m}^m]}[t, x(t), u_1^{(m)\theta_{\alpha_m}^m}, u_2^{(m)\theta_{\alpha_m}^m}, \cdots, u_n^{(m)\theta_{\alpha_m}^m}]\, e^{-r(t-t_\tau)} \right.$$

$$\left. + W_x^{[\theta_{\alpha_m}^m](m)}(t,x)\; f[t, x, u_1^{(m)\theta_{\alpha_m}^m}, u_2^{(m)\theta_{\alpha_m}^m}, \cdots, u_n^{(m)\theta_{\alpha_m}^m}] \right\}, \text{ and}$$

$$W^{[\theta_{\alpha_m}^m](m)}(T,x) = e^{-r(T-t_0)} \sum_{j=1}^{n} q^j(x);$$

$$-W_t^{[\theta_{\alpha_k}^k](k)}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{n}\Omega^{h\zeta}(t,x)W_{x^h x^\zeta}^{[\theta_{\alpha_k}^k](k)}(t,x)$$

$$= \max_{u_1^{\theta_{\alpha_k}}, u_2^{\theta_{\alpha_k}}, \cdots, u_2^{\theta_{\alpha_k}}} \left\{ \sum_{j=1}^{n} g^{[j,\theta_{\alpha_k}^k]}[t, x, u_1^{(k)\theta_{\alpha_k}^k}, u_2^{(k)\theta_{\alpha_k}^k}, \cdots, u_n^{(k)\theta_{\alpha_k}^k}]\, e^{-r(t-t_k)} \right.$$

$$\left. + W_x^{[\theta_{\alpha_k}^k](k)}(t,x)\; f[t, x, u_1^{(k)\theta_{\alpha_k}^k}, u_2^{(k)\theta_{\alpha_k}^k}, \cdots, u_n^{(k)\theta_{\alpha_k}^k}] \right\}, \text{ and}$$

$$W^{[\theta_{\alpha_k}^k](k)}(t_{k+1},x) = \sum_{a=1}^{\eta}\lambda_a\; W^{[\theta_a^{k+1}](k)}(t_{k+1},x), \quad \text{for } k \in \{0,1,2,\cdots,m-1\}. \qquad \blacksquare$$

References: *L. A. Petrosyan and D.W.K. Yeung: Subgame-consistent Cooperative Solutions in Randomly-furcating Stochastic Differential Games, International Journal of Mathematical and Computer Modelling (Special Issue on Lyapunov's Methods in Stability and Control), Vol. 45, June 2007, pp.1294-1307.*
L. A. Petrosyan and D.W.K. Yeung: Subgame Consistent Cooperation – A Comprehensive Treaties, Springer 2016.

Problem B9: ***Corresponding Problem of Theorem B9.***
Consider a class of randomly furcating cooperative stochastic differential game in which there are n players. The game interval is $[t_0, \mathrm{T}]$. When the game commences at t_0, the payoff structures of the players in the interval $[t_0, t_1)$ are known. In future instants of time t_k $(k = 1, 2, \cdots, m)$, where $t_0 < t_m < T \equiv t_{m+1}$, the payoff structures in the time interval $[t_k, t_{k+1})$ are affected by a series of random events Θ^k. In particular, Θ^k for $k \in \{1, 2, \cdots, m\}$, are independent and identically distributed random variables with range $\{\theta_1, \theta_2, ..., \theta_\eta\}$ and corresponding probabilities $\{\lambda_1, \lambda_2, ..., \lambda_\eta\}$. At time T a terminal value $q^i(x(T))$ will be given to player i. Player i seeks to maximize the expected payoff:

$$E_{t_0}\left\{ \int_{t_0}^{t_1} g^{[i,\theta_0^0]}[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)]\, e^{-r(s-t_0)} ds \right.$$

$$\left. + \sum_{h=1}^{m}\sum_{a_h=1}^{\eta}\lambda_{a_h}\int_{t_h}^{t_{h+1}} g^{[i,\theta_{a_h}^h]}[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)] e^{-r(s-t_0)} + e^{-r(T-t_0)} q^i(x(T)) \right\},$$

for $i \in \{1,2,\cdots,n\} \equiv N$,

where $x(s) \in X \subset R^{\kappa}$ is a vector of state variables, $\theta_{a_k}^{h} \in \{\theta_1, \theta_2, ..., \theta_\eta\}$ for $k \in \{1,2,\cdots,m\}$, $\theta_{a_0} = \theta_0^0$ is known at time t_0, r is the discount rate, $u_i \in U^i$ is the control of player i, and E_{t_0} denotes the expectation operator performed at time t_0. The payoffs of the players are transferable.

The state dynamics of the game is characterized by the vector-valued stochastic differential equations:

$$dx(s) = f[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)]ds + \sigma[s, x(s)]dz(s),$$
$$x(t_0) = x_0,$$

where $\sigma[s, x(s)]$ is a $\kappa \times \upsilon$ matrix and $z(s)$ is a υ-dimensional Wiener process and the initial state x_0 is given. Let $\Omega[s, x(s)] = \sigma[s, x(s)]\ \sigma[s, x(s)]^T$ denote the covariance matrix with its element in row h and column ζ denoted by $\Omega^{h\zeta}[s, x(s)]$. $u_i \in U_i \subset compR^{\ell}$ is the control vector of player i, for $i \in N$.

Theorem B10. (Subgame-consistent PDP for Random-horizon Stochastic Dynamic Cooperation under Uncertainty in Payoff Structures)

Consider the uncertain horizon randomly furcating cooperative stochastic dynamic game Problem B10 (see below) in which the players agree to maximize their joint expected payoff and share the cooperative gain according to the imputation $\xi^{(\sigma_k)}(k, x_k^*) = [\xi^{(\sigma_k)1}(k, x_k^*), \xi^{(\sigma_k)2}(k, x_k^*), \cdots, \xi^{(\sigma_k)n}(k, x_k^*)]$ along the cooperative trajectory given that $\theta_k^{\sigma_k}$ has occurred in stage k, for $\sigma_k \in \{1,2,\cdots,\eta_k\}$ and $k \in \{1,2,\cdots,T\}$. A Payoff Distribution Procedure (PDP) with a payment equaling

$$B_k^{(\sigma_k)i}(x_k^*) = \xi^{(\sigma_k)i}(k, x_k^*) - E_{\vartheta_k}\left\{ \frac{\varpi_k}{\sum_{\zeta=k}^{T}\varpi_\zeta}\ q_{k+1}^i[f_k(x_k^*, \psi_k^{(\sigma_k)*}(x_k^*)) + \vartheta_k] \right.$$
$$\left. + \frac{\sum_{\mu=k+1}^{T}\varpi_\mu}{\sum_{\zeta=k}^{T}\varpi_\zeta} \sum_{\sigma_{k+1}=1}^{\eta_k} \lambda_{k+1}^{\sigma_{k+1}}\ \xi^{(\sigma_{k+1})i}[k+1,\ f_k(x_k^*, \psi_k^{(\sigma_k)*}(x_k^*)) + \vartheta_k] \right\},$$

given to player $i \in N$ at stage $k \in \{1,2,\cdots,T\}$ if $\theta_k^{\sigma_k} \in \{\theta_k^1, \theta_k^2, \cdots, \theta_k^{\eta_k}\}$ occurs would lead to the realization of the imputation:
$\xi^{(\sigma_k)}(k, x_k^*) = [\xi^{(\sigma_k)1}(k, x_k^*), \xi^{(\sigma_k)2}(k, x_k^*), \cdots, \xi^{(\sigma_k)n}(k, x_k^*)]$, for $\sigma_k \in \{1,2,\cdots,\eta_k\}$ and $k \in \{1,2,\cdots,T\}$;
where
$\psi_k^{(\sigma_k)*}(x_k^*) = [\psi_k^{(\sigma_k)1*}(x_k^*), \psi_k^{(\sigma_k)2*}(x_k^*), \cdots, \psi_k^{(\sigma_k)n*}(x_k^*)]$, for $k \in \kappa$ and $i \in N$ is a set of controls that provides a group optimal solution to the Problem B10 yielding functions $W^{(\sigma_t)}(t, x)$, for $\sigma_t \in \{1,2,\cdots,\eta_t\}$ and $t \in \{1,2,\cdots,T\}$, such that the following recursive relations are satisfied:

$$W^{(\sigma_{T+1})}(T+1, x) = \sum_{j=1}^{n}\ q_{T+1}^j(x),$$

$$W^{(\sigma_T)}(T,x) = \max_{u_T^1,u_T^2,\cdots,u_T^n} E_{\vartheta_T}\left\{ \sum_{j=1}^{n} g_T^j(x,u_T^1,u_T^2,\cdots,u_T^n;\theta_T^{\sigma_T}) + W^{(\sigma_{T+1})}[T+1, f_T(x,u_T^1,u_T^2,\cdots,u_T^n)+\vartheta_T] \right\},$$

$$W^{(\sigma_\tau)}(\tau,x) = \max_{u_\tau^1,u_\tau^2,\cdots,u_\tau^n} E_{\vartheta_\tau}\left\{ \sum_{j=1}^{n} g_\tau^j(x,u_\tau^1,u_\tau^2,\cdots,u_\tau^n;\theta_\tau^{\sigma_\tau}) + \sum_{j=1}^{n} \frac{\varpi_\tau}{\sum_{\zeta=\tau}^{T}\varpi_\zeta} q_{\tau+1}^j[f_\tau(x,u_\tau^1,u_\tau^2,\cdots,u_\tau^n)+\vartheta_\tau] + \frac{\sum_{\zeta=\tau+1}^{T}\varpi_\zeta}{\sum_{\zeta=\tau}^{T}\varpi_\zeta} \sum_{\sigma_{\tau+1}=1}^{\eta_{\tau+1}} \lambda_{\tau+1}^{\sigma_{\tau+1}} W^{(\sigma_{\tau+1})}[\tau+1, f_\tau(x,u_\tau^1,u_\tau^2,\cdots,u_\tau^n)+\vartheta_\tau] \right\}$$

for $\tau \in \{1,2,\cdots,T-1\}$. ■

Reference: *D.W.K. Yeung and L.A. Petrosyan: Subgame Consistent Cooperative Solutions For Randomly Furcating Stochastic Dynamic Games With Uncertain Horizon. International Game Theory Review, Vol. 16, pp.1440012.01-1440012.29, 2014.*

Problem B10: ***Corresponding Problem of Theorem B10.***

Consider the $\hat{T}-$stage uncertain horizon randomly furcating cooperative stochastic dynamic game problem where $\hat{T}$ is a random variable with range $\{1,2,\cdots,T\}$ and corresponding probabilities $\{\varpi_1,\varpi_2,\cdots,\varpi_T\}$. Conditional upon the reaching of stage τ, the probability of the game would last up to stages $\tau,\tau+1,\cdots,T$ becomes respectively

$$\frac{\varpi_\tau}{\sum_{\zeta=\tau}^{T}\varpi_\zeta}, \frac{\varpi_{\tau+1}}{\sum_{\zeta=\tau}^{T}\varpi_\zeta}, \cdots, \frac{\varpi_T}{\sum_{\zeta=\tau}^{T}\varpi_\zeta}. \tag{1.1}$$

The state space of the game is $X \in R^m$ and the state dynamics of the game is characterized by the stochastic difference equation:

$$x_{k+1} = f_k(x_k,u_k^1,u_k^2,\cdots,u_k^n) + \vartheta_k, \tag{1.2}$$

for $k \in \{1,2,\cdots,T\}$ and $x_1 = x^0$,

where $u_k^i \in U^i \subset R^{m_i}$ is the control vector of player i at stage k, $x_k \in X$ is the state, and ϑ_k is a sequence of statistically independent random variables.

The payoff of player i at stage k is $g_k^i[x_k,u_k^1,u_k^2,\cdots,u_k^n;\theta_k]$ which is affected by a random variable θ_k. In particular, θ_k for $k \in \{1,2,\cdots,T\}$ are independent random variables with range $\{\theta_k^1,\theta_k^2,\cdots,\theta_k^{\eta_k}\}$ and corresponding probabilities $\{\lambda_k^1,\lambda_k^2,\cdots,\lambda_k^{\eta_k}\}$. In stage 1, it is known that θ_1 equals θ_1^1 with probability $\lambda_1^1 = 1$. When the game ends after stage $\hat{T}$, a terminal payment $q_{\hat{T}+1}^i(x_{\hat{T}+1})$ will be given to player i in stage $\hat{T}+1$.

The objective that player i seeks to maximize is

$$E_{\theta_1,\theta_2,\cdots,\theta_T;\vartheta_1,\vartheta_2,\cdots,\vartheta_T}\left\{ \sum_{\hat{T}=1}^{T}\varpi_{\hat{T}}\left[\sum_{k=1}^{\hat{T}} g_k^i[x_k,u_k^1,u_k^2,\cdots,u_k^n;\theta_k] + q^i(x_{\hat{T}+1}) \right] \right\},$$

for $i \in \{1,2,\cdots,n\} \equiv N$,

where $E_{\theta_1,\theta_2,\cdots,\theta_T;\vartheta_1,\vartheta_2,\cdots,\vartheta_T}$ is the expectation operation with respect to the random variables $\theta_1,\theta_2,\cdots,\theta_T$ and $\vartheta_1,\vartheta_2,\cdots,\vartheta_T$. Since there is no uncertainty in the payoff structure in stage $T+1$, we denote $\sigma_{T+1}=1$, $\theta_{T+1}^{\sigma_{T+1}}=\theta_{T+1}^1$ with probability $\lambda_{T+1}^{\sigma_{T+1}}=\lambda_{T+1}^1=1$ for notational convenience.

Theorem B11. (Subgame-consistent Solution Mechanism for Dynamic Cooperation under Non-transferrable Payoffs (NTU))

Consider the non-transferrable payoff/utility (NTU) cooperative dynamic game Problem B11 (see below) in which the players agree to use a set of payoff weights $\{\hat{\alpha}_k = (\hat{\alpha}_k^1,\hat{\alpha}_k^2,\cdots,\hat{\alpha}_k^n)$, for $k\in\kappa\}$ for joint maximization of the weighted joint payoff so that the imputation $\xi^i(k,x_k^*)$ for player $i\in N$ in stage $k\in\kappa$ along the cooperative trajectory $\{x_k^*\}_{k=1}^T$ can be achieved.

A set of payoff weights $\{\hat{\alpha}_k = (\hat{\alpha}_k^1,\hat{\alpha}_k^2,\cdots,\hat{\alpha}_k^n)$, for $k\in\kappa\}$ and a set of strategies $\{\psi_k^{(\hat{\alpha}_k)i}(x)$, for $k\in\kappa$ and $i\in N\}$ provides a subgame consistent solution to the NTU cooperative dynamic game Problem B11 if there exist functions $W^{(\hat{\alpha}_k)}(k,x)$ and $W^{(\hat{\alpha}_k)i}(k,x)$, for $i\in N$ $k\in\kappa$, which satisfy the following recursive relations:

$$W^{(\hat{\alpha}_{T+1})i}(T+1,x) = q^i(x_{T+1}),$$

$$W^{(\hat{\alpha}_k)}(k,x) = \max_{u_k^1,u_k^2,\cdots,u_k^n}\left\{ \sum_{j=1}^{n}\hat{\alpha}^j g_k^j(x,u_k^1,u_k^2,\cdots,u_k^n) + \sum_{j=1}^{n}\hat{\alpha}_k^j W^{(\hat{\alpha}_{k+1})j}[k+1, f_k(x,u_k^1,u_k^2,\cdots,u_k^n)] \right\};$$

$$W^{(\hat{\alpha}_k)i}(k,x) = g_k^j(x,\psi_k^{(\hat{\alpha}_k)1}(x),\psi_k^{(\hat{\alpha}_k)2}(x),\cdots,\psi_k^{(\hat{\alpha}_k)n}(x)) + W^{(\hat{\alpha}_{k+1})i}[k+1, f_k(x,\psi_k^{(\hat{\alpha}_k)1}(x),\psi_k^{(\hat{\alpha}_k)2}(x),\cdots,\psi_k^{(\hat{\alpha}_k)n}(x))],$$

for $i\in N$ and $k\in\kappa$;

and imputation $\xi^i(k,x_k^*)$ for player $i\in N$ in stage $k\in\kappa$,

where the value function $W^{(\hat{\alpha}_k)i}(k,x)$ is the payoff for player $i\in N$ in stage $k\in\kappa$ under cooperation. ■

Reference: *D.W.K. Yeung and L.A. Petrosyan: Subgame Consistent Cooperative Solution for NTU Dynamic Games via Variable Weights, forthcoming in Automatica, 2015.*

Problem B11: ***Corresponding Problem of Theorem B11.***

Consider the general $T-$stage $n-$person dynamic game with initial state x_1^0. The state space of the game is $X\in R^m$ and the state dynamics of the game is characterized by the difference equation:

$$x_{k+1} = f_k(x_k,u_k^1,u_k^2,\cdots,u_k^n),$$

for $k \in \{1,2,\cdots,T\} \equiv \kappa$ and $x_1 = x_1^0$,
where $u_k^i \in R^{m_i}$ is the control vector of player i at stage k, and $x_k \in X$ is the state of the game. The payoff that player i seeks to maximize is

$$\sum_{k=1}^{T} g_k^i(x_k, u_k^1, u_k^2, \cdots, u_k^n) + q^i(x_{T+1}),$$

for $i \in \{1,2,\cdots,n\} \equiv N$,
where $q^i(x_{T+1})$ is the terminal payoff that player i will received in stage $T+1$.

The payoffs of the players are not transferable. The players agree to use a set of payoff weights $\{\hat{\alpha}_k = (\hat{\alpha}_k^1, \hat{\alpha}_k^2, \cdots, \hat{\alpha}_k^n)$, for $k \in \kappa\}$ for joint maximization of the weighted joint payoff so that the imputation $\xi^i(k, x_k^*)$ for player $i \in N$ in stage $k \in \kappa$ along the cooperative trajectory $\{x_k^*\}_{k=1}^T$ can be achieved.

Theorem B12. (Subgame-consistent Solution Mechanism for Stochastic Dynamic Cooperation under Non-transferrable Payoffs (NTU))

Consider the non-transferrable payoff/utility (NTU) cooperative stochastic dynamic game Problem B12 (see below) in which the players agree to use a set of payoff weights $\{\hat{\alpha}_k = (\hat{\alpha}_k^1, \hat{\alpha}_k^2, \cdots, \hat{\alpha}_k^n)$, for $k \in \kappa\}$ for joint maximization of the expected weighted joint payoff so that the imputation $\xi^i(k, x_k^*)$ for player $i \in N$ in stage $k \in \kappa$ along the cooperative trajectory $\{x_k^*\}_{k=1}^T$ can be achieved.

A set of payoff weights $\{\hat{\alpha}_k = (\hat{\alpha}_k^1, \hat{\alpha}_k^2, \cdots, \hat{\alpha}_k^n)$, for $k \in \kappa\}$ and a set of strategies $\{\psi_k^{(\hat{\alpha}_k)i}(x)$, for $k \in \kappa$ and $i \in N\}$ provides a subgame consistent solution to the NTU cooperative dynamic game Problem B12 if there exist functions $W^{(\hat{\alpha}_k)}(k, x)$ and $W^{(\hat{\alpha}_k)i}(k, x)$, for $i \in N$ $k \in \kappa$, which satisfy the following recursive relations:

$$W^{(\hat{\alpha}_{T+1})i}(T+1, x) = q^i(x_{T+1}),$$

$$W^{(\hat{\alpha}_k)}(k, x) = \max_{u_k^1, u_k^2, \cdots, u_k^n} \Bigg\{ E_{\theta_k}\Bigg[\sum_{j=1}^{n} \hat{\alpha}^j g_k^j(x, u_k^1, u_k^2, \cdots, u_k^n) + \sum_{j=1}^{n} \hat{\alpha}_k^j W^{(\hat{\alpha}_{k+1})j}[k+1, f_k(x, u_k^1, u_k^2, \cdots, u_k^n) + G_k(x)\theta_k] \Bigg] \Bigg\};$$

$$W^{(\hat{\alpha}_k)i}(k, x) = E_{\theta_k}\Bigg\{ g_k^j(x, \psi_k^{(\hat{\alpha}_k)1}(x), \psi_k^{(\hat{\alpha}_k)2}(x), \cdots, \psi_k^{(\hat{\alpha}_k)n}(x)) + W^{(\hat{\alpha}_{k+1})i}[k+1, f_k(x, \psi_k^{(\hat{\alpha}_k)1}(x), \psi_k^{(\hat{\alpha}_k)2}(x), \cdots, \psi_k^{(\hat{\alpha}_k)n}(x)) + G_k(x)\theta_k] \Bigg\},$$

for $i \in N$ and $k \in \kappa$;
and imputation $\xi^i(k, x_k^*)$ for player $i \in N$ in stage $k \in \kappa$,

where the value function $W^{(\hat{\alpha}_k)i}(k, x)$ is the expected payoff for player $i \in N$ in stage $k \in \kappa$ under cooperation. ■

Reference: *D.W.K. Yeung and L.A. Petrosyan: On Subgame Consistent Solution for NTU Cooperative Stochastic Dynamic Games, paper presented at European Meeting on Game Theory (SING11-GTM2015) at St Petersburg, July 8-10, 2015.*

Problem B12: ***Corresponding Problem of Theorem B12.***

Consider the NTU cooperative stochastic dynamic game with initial state x_1^0. The state space of the game is $X \in R^m$ and the state dynamics of the game is characterized by the stochastic difference equation:

$$x_{k+1} = f_k\,(x_k, u_k^1, u_k^2, \cdots, u_k^n) + G_k(x_k)\theta_k\,,$$

for $k \in \{1,2,\cdots,T\} \equiv \kappa$ and $x_1 = x_1^0$,

where $u_k^i \in R^{m_i}$ is the control vector of player i at stage k, and $x_k \in X$ is the state of the game and θ_k is a set of independent random variable. The payoff that player i seeks to maximize is

$$E_{\theta_1,\theta_2,\cdots,\theta_T}\left\{ \sum_{\zeta=1}^{T} g_\zeta^i[x_\zeta, u_\zeta^1, u_\zeta^2, \cdots, u_\zeta^n, x_{\zeta+1}] + q^i(x_{T+1}) \right\},$$

for $i \in \{1,2,\cdots,n\} \equiv N$,

where $q^i(x_{T+1})$ is the terminal payoff that player i will received in stage $T+1$, and $E_{\theta_1,\theta_2,\cdots,\theta_T}$ is the expectation operation with respect to the statistics of $\theta_1, \theta_2, \cdots, \theta_T$.

The payoffs of the players are not transferable. The payoffs of the players are not transferable. The players agree to use a set of payoff weights $\{\hat{\alpha}_k = (\hat{\alpha}_k^1, \hat{\alpha}_k^2, \cdots, \hat{\alpha}_k^n)$, for $k \in \kappa\}$ for joint maximization of the expected weighted joint payoff so that the imputation $\xi^i(k, x_k^*)$ for player $i \in N$ in stage $k \in \kappa$ along the cooperative trajectory $\{x_k^*\}_{k=1}^T$ can be achieved.

Theorem B13. Hamilton-Jacobi-Bellman Equations for Dynamic Games with Durable Controls

Let $V^i(k, x; \underline{u}_{k-}^{**})$ be the feedback Nash equilibrium payoff of player i in the noncooperative game (3.1)-(3.2), then the function $V^i(k, x; \underline{u}_{k-}^{**})$ satisfies the following recursive equations:

$$V^i(T+1, x; \underline{u}_{(T+1)-}^{**}) = q_{T+1}^i(x; \underline{u}_{(T+1)-}^{**})\,;$$

$$V^i(k, x; \underline{u}_{k-}^{**}) = \max_{u_k^i}\Big\{ g_k^i(x, u_k^i, \underline{u}_k^{**(\neq i)}; \underline{u}_{k-}^{**})$$

$$+ V^i[k+1, f_k\,(x, u_k^i, \underline{u}_k^{**(\neq i)}; \underline{u}_{k-}^{**}); u_k^i, u_{k-\cap(k-1)-}^{**i}, \underline{u}_{(k+1)-}^{**(\neq i)}] \Big\},$$

for $k \in \{1,2,\cdots,T\}$ and $i \in N$,

where $\underline{u}_k^{**} = (u_k^{**1}, u_k^{**2}, \cdots, u_k^{**n})$ are the corresponding Nash equilibrium strategies,

$\underline{u}_k^{**(\neq i)} = (u_k^{**1}, u_k^{**2}, \cdots, u_k^{**i-1}, u_k^{**i+1}, \cdots, u_k^{**n})$,

$\underline{u}_{k-}^{**(\neq i)} = (u_{k-}^{**1}, u_{k-}^{**2}, \cdots, u_{k-}^{**i-1}, u_{k-}^{**i+1}, \cdots, u_{k-}^{**n})$, and

$u_{k-\cap(k-1)-}^{**i}$ are the elements in the intersection of the set of controls $u_{(k-1)-}^{**i}$ and the set of controls u_{k-}^{**i}. ■

References: *D.W.K. Yeung, L.A. Petrosyan (2020): Cooperative Dynamic Games with Durable Controls: Theory and Application, Dynamic Games and Applications, Doi:10.1007/s13235-019-00336-w.*
9(2), 550-567, 2019, https://doi.org/10.1007/s13235-018-0266-6.
D.W.K. Yeung, L.A. Petrosyan (2019): Cooperative Dynamic Games with Control Lags, Dynamic Games and Applications, 9(2), 550-567, https://doi.org/10.1007/s13235-018-0266-6.

Theorem B14. Subgame-consistent PDP for Cooperative Dynamic Games with Durable Controls

The agreed-upon imputation $\xi(k, x_k^*; \underline{u}_{k-}^*)$, for $k \in \{1,2,\cdots,T\}$ along the cooperative trajectory $\{x_k^*\}_{k=1}^T$, can be realized by a payment

$$B_k^i(x_k^*; \underline{u}_{k-}^*) = \left[\xi^i(k, x_k^*; \underline{u}_{k-}^*) - \xi^i\left(k+1,\ f_k\,(x_k^*, \underline{u}_k^*; \underline{u}_{k-}^*)\ ; \underline{u}_{(k+1)-}^* \right) \right]$$

given to player $i \in N$ at stage $k \in \{1,2,\cdots,T\}$. ■

References: *D.W.K. Yeung, L.A. Petrosyan (2020): Cooperative Dynamic Games with Durable Controls: Theory and Application, Dynamic Games and Applications, Doi:10.1007/s13235-019-00336-w.*
9(2), 550-567, 2019, https://doi.org/10.1007/s13235-018-0266-6.

D.W.K. Yeung, L.A. Petrosyan (2019): Cooperative Dynamic Games with Control Lags, Dynamic Games and Applications, 9(2), 550-567, https://doi.org/10.1007/s13235-018-0266-6.

Part C: Identities and Equations in Economics

C1. Inter-temporal Roy's Identity

$$\frac{\partial v^\ell(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_h^j} \div \frac{\partial v^\ell(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial W_\ell^0}$$
$$\equiv -(1+r)^{-(h-\ell)} \varphi_h^j(W_h^0, p_h, p_{h+1}, \cdots, p_T);$$

or in an alternative form

$$\frac{\partial v^{\ell}(W_{\ell}^{0}, p_{\ell}, p_{\ell+1}, \cdots, p_{T})}{\partial p_{h}^{j}} \div \delta_{\ell+1}^{h} \frac{\partial v^{h}(W_{h}^{0}, p_{h}, p_{h+1}, \cdots, p_{T})}{\partial W_{h}^{0}}$$

$$\equiv -\varphi_{h}^{j}(W_{h}^{0}, p_{h}, p_{h+1}, \cdots, p_{T});$$

for $\ell \in \{1,2,\cdots,T\}$, $h \in \{\ell, \ell+1, \cdots, T\}$ and $j \in \{1,2,\cdots,n_h\}$,

where

$$W_{\ell} = W_{\ell}^{0},$$

$$W_{\ell+1}^{0} = (1+r)(W_{\ell}^{0} - p_{\ell}\varphi_{\ell}) + Y_{\ell+1},$$

$$W_{\ell+2}^{0} = (1+r)(W_{\ell+1}^{0} - p_{\ell+1}\varphi_{\ell+1}) + Y_{\ell+2},$$

$$\vdots \qquad \vdots$$

$$W_{h}^{0} = (1+r)(W_{h-1}^{0} - p_{h-1}\varphi_{h-1}) + Y_{h}. \qquad \blacksquare$$

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*
D.W.K. Yeung: Optimal Consumption under an Uncertain Inter-temporal Budget: Stochastic Dynamic Slutsky Equations, Vestnik St Petersburg University: Mathematics (Springer), Vol. 10, 2013, pp.121-141

Problem C1: ***Corresponding Problem of Theorem C1.***

The inter-temporal Roy's identity is derived from the consumer problem in which the consumer maximizes his inter-temporal utility

$$u^{1}(x_{1}^{1}, x_{1}^{2}, \cdots, x_{1}^{n_1}) + \sum_{k=2}^{T} \delta_{2}^{k} u^{k}(x_{k}^{1}, x_{k}^{2}, \cdots, x_{k}^{n_k})$$

$$= u^{1}(x_{1}) + \sum_{k=2}^{T} \delta_{2}^{k} u^{k}(x_{k}) = \sum_{k=1}^{T} \delta_{1}^{k} u^{k}(x_{k})$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1} = W_{k} - \sum_{h=1}^{\eta_k} p_{k}^{h} x_{k}^{h} + r(W_{k} - \sum_{h=1}^{\eta_k} p_{k}^{h} x_{k}^{h}) + Y_{k+1}, \qquad W_{1} = W_{1}^{0},$$

where

$x_{k} = (x_{k}^{1}, x_{k}^{2}, \cdots, x_{k}^{n_k})$ is the vector of quantities of goods consumed in period k, $p_{k} = (p_{k}^{1}, p_{k}^{2}, \cdots, p_{k}^{n_k})$ is price vector, r is the interest rate, Y_{k} is the income that the consumer will receive in period k, $\delta_{2}^{k} = \left(\prod_{c=2}^{k} \beta_{c} \right)$ is the discount factor with β_{τ} being the consumer's subjective one-period discount factor for the duration from period $\tau - 1$ to period τ, $\beta_{1} = 1$ for the discount factor in the initial period 1 and $\delta_{1}^{k} = \left(\prod_{c=1}^{k} \beta_{c} \right) = \delta_{2}^{k} = \left(\prod_{c=2}^{k} \beta_{c} \right)$. The period k utility function $u^{k}(x_{k}^{1}, x_{k}^{2}, \cdots, x_{k}^{n_k})$ is continuously differentiable and quasi-concave yielding convex level (indifference) curves. The time preference factor is embodied in the utility function. The time preference factor is embodied in the utility function. The amount of unconsumed wealth $W_{k} - p_{k}x_{k}$ in period k will generate an interest income $r(W_{k} - p_{k}x_{k})$ in period $k+1$.

In addition, $v^{\ell}(W_{\ell}^{0}, p_{\ell}, p_{\ell+1}, \cdots, p_T)$ is the intertemporal indirect utility at period ℓ, and $\varphi_h^j(W_h^0, p_h, p_{h+1}, \cdots, p_T)$ is the ordinary demand function of commodity j in period h.

C2. Inter-temporal Roy's Identity under Stochastic Income

$$\frac{\partial v^{\ell}(W_{\ell}^{0}, p_{\ell}, p_{\ell+1}, \cdots, p_T)}{\partial p_{\ell}^{j}} \div \frac{\partial v^{\ell}(W_{\ell}^{0}, p_{\ell}, p_{\ell+1}, \cdots, p_T)}{\partial W_{\ell}^{0}}$$
$$\equiv -\varphi_{\ell}^{j}(W_{\ell}^{0}, p_{\ell}, p_{\ell+1}, \cdots, p_T), \text{ for } j \in \{1,2,\cdots,n_{\ell}\};$$

$$\frac{\partial v^{\ell}(W_{\ell}^{0}, p_{\ell}, p_{\ell+1}, \cdots, p_T)}{\partial p_{h}^{j}} \div \frac{\partial v^{\ell}(W_{\ell}^{0}, p_{\ell}, p_{\ell+1}, \cdots, p_T)}{\partial W_{\ell}^{0}}$$
$$\equiv -\sum_{j_{\ell+1}=1}^{m_{\ell+1}} \lambda_{\ell+1}^{j_{\ell+1}} \sum_{j_{\ell+2}=1}^{m_{\ell+2}} \lambda_{\ell+2}^{j_{\ell+2}} \cdots \sum_{j_h=1}^{m_h} \lambda_h^{j_h} \; \delta_{\ell+1}^{h} \, \frac{\partial v^h(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+1}^{j_{\ell+2}}\cdots\theta_h^{j_h}}, p)}{\partial W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+1}^{j_{\ell+2}}\cdots\theta_h^{j_h}}}$$
$$\times \frac{\varphi_h^j(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h}}, p)(1+r)^{-(h-\ell)}}{\sum_{w_{\ell+1}=1}^{m_{\ell+1}} \lambda_{\ell+1}^{w_{\ell+1}} \sum_{w_{\ell+2}=1}^{m_{\ell+2}} \lambda_{\ell+2}^{w_{\ell+2}} \cdots \sum_{w_h=1}^{m_h} \lambda_h^{w_h} \delta_{\ell+1}^{h} \frac{\partial v^h(W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_{h-1}^{w_{h-1}}}, p)}{\partial W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_{h-1}^{w_{h-1}}}}},$$

for $\ell \in \{1,2,\cdots,T\}$ $h \in \{\ell+1, \ell+2, \cdots, T\}$ and $j \in \{1,2,\cdots,n_h\}$,
and $v^h(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h}}, p)$ is the short form for $v^h(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h}}, p_h, p_{h+1}, \cdots, p_T)$,
where

$$W_{\ell} = W_{\ell}^{0},$$
$$W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}}} = (1+r)[W_{\ell}^{0} - p_{\ell}\varphi_{\ell}(W_{\ell}^{0}, p)] + \theta_{\ell+1}^{j_{\ell+1}},$$
$$W_{\ell+2}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}} = (1+r)[W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}}} - p_{\ell+1}\varphi_{\ell+1}(W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}}}, p)] + \theta_{\ell+2}^{j_{\ell+2}},$$
$$\vdots \qquad\qquad \vdots$$
$$W_T^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_T^{j_T}} = (1+r)[W_{T-1}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T-1}^{j_{T-1}}} - p_{T-1}\varphi_{T-1}(W_{T-1}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T-1}^{j_{T-1}}})] + \theta_T^{j_T}. \qquad ■$$

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*
D.W.K. Yeung: Optimal Consumption under an Uncertain Inter-temporal Budget: Stochastic Dynamic Slutsky Equations, Vestnik St Petersburg University: Mathematics (Springer), Vol. 10, 2013, pp.121-141.

Problem C2: ***Corresponding Problem of Theorem C2.***
The inter-temporal Roy's identity under stochastic income is derived from the consumer problem in which the consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ \sum_{k=1}^{T} \delta_1^k u^k(x_k^1, x_k^2, \cdots, x_k^{n_k}) \right\} = E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ \sum_{k=1}^{T} \delta_1^k u^k(x_k) \right\}$$

subject to the budget constraint characterized by the stochastic wealth dynamics

$$W_{k+1} = (1+r)(W_k - p_k x_k) + \theta_{k+1}, \qquad W_1 = W_1^0,$$

where

θ_k is the random income that the consumer will receive in period k; and θ_k, for $k \in \{2, \cdots, T\}$, is a set of statistically independent random variables, and $E_{\theta_1, \theta_2, \cdots, \theta_T}$ is the expectation operation with respect to the statistics of $\theta_2, \theta_3, \cdots, \theta_T$. The random variable θ_k has a non-negative range $\{\theta_k^1, \theta_k^2, \cdots, \theta_k^{m_k}\}$ with corresponding probabilities $\{\lambda_k^1, \lambda_k^2, \cdots, \lambda_k^{m_k}\}$, for $k \in \{2, \cdots, T\}$.

C3. Inter-temporal Roy's Identity under Stochastic Life-span

$$\frac{\partial v^\ell(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_h^j} \div \frac{\partial v^\ell(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial W_\ell^0}$$
$$\equiv -(1+r)^{-(h-\ell)} \varphi_h^j(W_h^0, p_h, p_{h+1}, \cdots, p_T);$$

or in an alternatively form:

$$\frac{\partial v^\ell(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_h^j} \div \delta_{\ell+1}^h \frac{\partial v^h(W_h^0, p_h, p_{h+1}, \cdots, p_T)}{\partial W_h^0}$$
$$\equiv -\frac{\sum_{\zeta=h}^{T} \gamma_\zeta}{\sum_{\zeta=\ell}^{T} \gamma_\zeta} \varphi_h^j(W_h^0, p_h, p_{h+1}, \cdots, p_T);$$

for $\ell \in \{1,2,\cdots,T\}$ $h \in \{\ell, \ell+1, \cdots, T\}$ and $j \in \{1,2,\cdots,n_h\}$,

where

$$W_\ell = W_\ell^0,$$
$$W_{\ell+1}^0 = (1+r)(W_\ell^0 - p_\ell \varphi_\ell) + Y_{\ell+1},$$
$$W_{\ell+2}^0 = (1+r)(W_{\ell+1}^0 - p_{\ell+1} \varphi_{\ell+1}) + Y_{\ell+2},$$
$$\vdots \qquad \vdots$$
$$W_h^0 = (1+r)(W_{h-1}^0 - p_{h-1} \varphi_{h-1}) + Y_h. \qquad \blacksquare$$

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*
D.W.K. Yeung: Optimal Consumption under Uncertainties: Random Horizon Stochastic Dynamic Roy's Identity and Slutsky Equation, Applied Mathematics, Vol.5, 2014, pp.263-284.

Problem C3: ***Corresponding Problem of Theorem C3.***

The inter-temporal Roy's identity under stochastic life-span is derived from the consumer problem in which the consumer's life-span involves $\hat{T}$ periods where $\hat{T}$ is a random variable with range $\{1,2,\cdots,T\}$ and corresponding probabilities $\{\gamma_1, \gamma_2, \cdots, \gamma_T\}$. Conditional upon the reaching of period τ, the probability of the consumer's life-span would last up to periods $\tau, \tau+1, \cdots, T$ becomes respectively

$$\frac{\gamma_\tau}{\sum_{\zeta=\tau}^{T} \gamma_\zeta}, \frac{\gamma_{\tau+1}}{\sum_{\zeta=\tau}^{T} \gamma_\zeta}, \cdots, \frac{\gamma_T}{\sum_{\zeta=\tau}^{T} \gamma_\zeta}.$$

The consumer maximizes his expected inter-temporal utility

$$\sum_{\hat{T}=1}^{T}\gamma_{\hat{T}}\ \sum_{k=1}^{\hat{T}}\delta_1^k u^k(x_k),$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1} = W_k - \sum_{h=1}^{n_k} p_k^h x_k^h + r(W_k - \sum_{h=1}^{n_k} p_k^h x_k^h) + Y_{k+1}, \qquad W_1 = W_1^0.$$

where

r is the interest rate, Y_k is the income that the consumer will receive in period k.

C4. Inter-temporal Roy's Identity under Stochastic Income and Life-span

$$\frac{\partial v^\ell(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_\ell^j} \div \frac{\partial v^\ell(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial W_\ell^0}$$

$$\equiv -\varphi_\ell^j(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T), \text{ for } j \in \{1,2,\cdots,n_\ell\},$$

$$\frac{\partial v^\ell(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_h^k} \div \frac{\partial v^\ell(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial W_\ell^0}$$

$$\equiv -\sum_{j_{\ell+1}=1}^{m_{\ell+1}}\lambda_{\ell+1}^{j_{\ell+1}}\sum_{j_{\ell+2}=1}^{m_{\ell+2}}\lambda_{\ell+2}^{j_{\ell+2}}\cdots\sum_{j_h=1}^{m_h}\lambda_h^{j_h}\ \delta_{\ell+1}^h\ \frac{\partial v^h(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h}}, p)}{\partial W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h}}}\ \varphi_h^k(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h}}, p)\ (1+r)^{-(h-\ell)}$$

$$\div\left[\ \sum_{w_{\ell+1}=1}^{m_{\ell+1}}\lambda_{\ell+1}^{w_{\ell+1}}\sum_{w_{\ell+2}=1}^{m_{\ell+2}}\lambda_{\ell+2}^{w_{\ell+2}}\cdots\sum_{w_h=1}^{m_h}\lambda_h^{w_h}\ \delta_{\ell+1}^h\ \frac{\partial v^h(W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_h^{w_h}}, p)}{\partial W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_h^{w_h}}}\ \right];$$

or in an alternative form:

$$\frac{\partial v^\ell(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_h^k}$$

$$\equiv -\frac{\sum_{\zeta=h}^{T}\gamma_\zeta}{\sum_{\zeta=\ell}^{T}\gamma_\zeta}\ \sum_{j_{\ell+1}=1}^{m_{\ell+1}}\lambda_{\ell+1}^{j_{\ell+1}}\sum_{j_{\ell+2}=1}^{m_{\ell+2}}\lambda_{\ell+2}^{j_{\ell+2}}\cdots\sum_{j_h=1}^{m_h}\lambda_h^{j_h}\ \delta_{\ell+1}^h\ \frac{\partial v^h(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h}}, p)}{\partial W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h}}}\ \varphi_h^k(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h}}, p);$$

for $\ell \in \{1,2,\cdots,T\}$, $h \in \{\ell+1, \ell+2, \cdots, T\}$ and $k \in \{1,2,\cdots,n_h\}$,

and $v^h(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h}}, p)$ is the short form for $v^h(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h}}, p_h, p_{h+1}, \cdots, p_T)$,

where

$$W_\ell = W_\ell^0,$$

$$W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}}} = (1+r)[W_\ell^0 - p_\ell\varphi_\ell(W_\ell^0, p)] + \theta_{\ell+1}^{j_{\ell+1}},$$

$$W_{\ell+2}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}} = (1+r)[W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}}} - p_{\ell+1}\varphi_{\ell+1}(W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}}}, p)] + \theta_{\ell+2}^{j_{\ell+2}},$$

$$\vdots \qquad\qquad \vdots$$

$$W_T^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_T^{j_T}} = (1+r)[W_{T-1}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T-1}^{j_{T-1}}} - p_{T-1}\varphi_{T-1}(W_{T-1}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T-1}^{j_{T-1}}})] + \theta_T^{j_T}. \quad \blacksquare$$

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*
D.W.K. Yeung: Optimal Consumption under Uncertainties: Random Horizon Stochastic Dynamic Roy's Identity and Slutsky Equation, Applied Mathematics, Vol.5,

2014, pp.263-284.

Problem C4: ***Corresponding Problem of Theorem C4.***

The inter-temporal Roy's identity under stochastic income and life-span is derived from the consumer problem in which the consumer's life-span involves $\hat{T}$ periods where $\hat{T}$ is a random variable with range $\{1,2,\cdots,T\}$ and corresponding probabilities $\{\gamma_1,\gamma_2,\cdots,\gamma_T\}$. Conditional upon the reaching of period τ, the probability of the consumer's life-span would last up to periods $\tau,\tau+1,\cdots,T$ becomes respectively

$$\frac{\gamma_\tau}{\sum_{\zeta=\tau}^{T}\gamma_\zeta},\ \frac{\gamma_{\tau+1}}{\sum_{\zeta=\tau}^{T}\gamma_\zeta},\cdots,\frac{\gamma_T}{\sum_{\zeta=\tau}^{T}\gamma_\zeta}.$$

The consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3,\cdots,\theta_{T+1}}\left\{\ \sum_{\hat{T}=1}^{T}\gamma_{\hat{T}}\ \sum_{k=1}^{\hat{T}}\delta_1^k u^k(x_k)\ \right\},$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1}=W_k-\sum_{h=1}^{n_k}p_k^h x_k^h+r(W_k-\sum_{h=1}^{n_k}p_k^h x_k^h)+\theta_{k+1},\qquad W_1=W_1^0.$$

where

θ_k is the random income that the consumer will receive in period k; and θ_k, for $k\in\{2,\cdots,T\}$, is a set of statistically independent random variables, and $E_{\theta_1,\theta_2,\cdots,\theta_T}$ is the expectation operation with respect to the statistics of $\theta_2,\theta_3,\cdots,\theta_T$. The random variable θ_k has a non-negative range $\{\theta_k^1,\theta_k^2,\cdots,\theta_k^{m_k}\}$ with corresponding probabilities $\{\lambda_k^1,\lambda_k^2,\cdots,\lambda_k^{m_k}\}$, for $k\in\{2,\cdots,T\}$.

C5. Inter-temporal Roy's Identity under Stochastic Preferences

$$\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial p_\ell^j}\div\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial W_\ell^0}\equiv-\varphi_\ell^{(\upsilon_\ell)j}(W_\ell^0,p),\ \text{for}\ j\in\{1,2,\cdots,n_\ell\},$$

$$\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial p_h^k}\div\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial W_\ell^0}\equiv-\sum_{\upsilon_{\ell+1}=1}^{\overline{m}_{\ell+1}}\rho_{\ell+1}^{\upsilon_{\ell+1}}\sum_{\upsilon_{\ell+2}=1}^{\overline{m}_{\ell+2}}\rho_{\ell+2}^{\upsilon_{\ell+2}}\cdots$$

$$\cdots\sum_{\upsilon_h=1}^{\overline{m}_h}\rho_h^{\upsilon_h}\ \delta_{\ell+1}^h\ \frac{\partial v^{h(\upsilon_h)}(W_h^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}},p)}{\partial W_h^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}}}\ \varphi_h^{(\upsilon_h)k}(W_h^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}},p)\ (1+r)^{-(h-\ell)}$$

$$\div\left[\ \sum_{\varpi_{\ell+1}=1}^{\overline{m}_{\ell+1}}\rho_{\ell+1}^{\varpi_{\ell+1}}\sum_{\varpi_{\ell+2}=1}^{\overline{m}_{\ell+2}}\rho_{\ell+2}^{\varpi_{\ell+2}}\cdots\sum_{\varpi_h=1}^{\overline{m}_h}\rho_h^{\varpi_h}\ \delta_{\ell+1}^h\ \frac{\partial v^{h(\varpi_h)}(W_h^{\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}},p)}{\partial W_h^{\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}}}\ \right];$$

or in an alternative form

$$\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial W_\ell^0}\equiv-\sum_{\upsilon_{\ell+1}=1}^{\overline{m}_{\ell+1}}\rho_{\ell+1}^{\upsilon_{\ell+1}}\sum_{\upsilon_{\ell+2}=1}^{\overline{m}_{\ell+2}}\rho_{\ell+2}^{\upsilon_{\ell+2}}\cdots$$

$$\cdots\sum_{\upsilon_{h+1}=1}^{\overline{m}_{h+1}}\rho_h^{\upsilon_h}\ \delta_{\ell+1}^{h+1}\ \frac{\partial v^{h(\upsilon_h)}(W_h^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}},p)}{\partial W_h^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}}}\ (1+r)^{h-\ell};$$

for $\ell \in \{1,2,\cdots,T\}$, $h \in \{\ell+1, \ell+2,\cdots,T\}$, $k \in \{1,2,\cdots,n_h\}$ and $\upsilon_\ell \in \{1,2,\cdots,\overline{m}_\ell\}$,
where

$W_\ell = W_\ell^0$,

$W_{\ell+1}^{\upsilon_\ell} = (1+r)[W_\ell^0 - p_\ell \varphi^{(\upsilon_\ell)}{}_\ell(W_\ell^0, p)] + Y_{\ell+1}$,

$W_{\ell+2}^{\upsilon_\ell \upsilon_{\ell+1}} = (1+r)[W_{\ell+1}^{\upsilon_\ell} - p_{\ell+1}\varphi_{\ell+1}^{(\upsilon_{\ell+1})}(W_{\ell+1}^{\upsilon_\ell}, p)] + Y_{\ell+2}$,

if preference is $u^{\ell+1(\upsilon_{\ell+1})}(x_{\ell+1})$ in period $\ell+1$;

$W_{\ell+3}^{\upsilon_\ell \upsilon_{\ell+1} \upsilon_{\ell+2}} = (1+r)[W_{\ell+2}^{\upsilon_\ell \upsilon_{\ell+1}} - p_{\ell+2}\varphi_{\ell+2}^{(\upsilon_{\ell+1})}(W_{\ell+2}^{\upsilon_\ell \upsilon_{\ell+1}}, p)] + Y_{\ell+3}$,

if preference is $u^{\ell+2(\upsilon_{\ell+2})}(x_{\ell+2})$ in period $\ell+2$;

$\vdots \qquad \vdots$

$W_T^{\upsilon_\ell, \upsilon_{\ell+1} \cdots \upsilon_{T-1}} = (1+r)[W_{T-1}^{\upsilon_\ell \upsilon_{\ell+1} \cdots \upsilon_{T-2}} - p_{T-1}\varphi_{T-1}^{(\upsilon_{T-1})}(W_{T-1}^{\upsilon_\ell \upsilon_{\ell+1} \cdots \upsilon_{T-2}})] + Y_T$;

if preference is $u^{T-1(\upsilon_{\ell+2})}(x_{T-1})$ in period $T-1$;

$W_{T+1}^{\upsilon_\ell, \upsilon_{\ell+1} \cdots \upsilon_T} = (1+r)[W_T^{\upsilon_\ell \upsilon_{\ell+1} \cdots \upsilon_{T-1}} - p_T \varphi_T^{(\upsilon\ )}(W_T^{\upsilon_\ell \upsilon_{\ell+1} \cdots \upsilon_{T-1}})] + Y_{T+1} = 0$. ■

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*
D.W.K. Yeung: Random Horizon Stocahstic dynamic Slutsky Equation under Preference Uncertainty, Applied Mathematical Sciences, Vol. 8, 2014, pp.7311-7340.

Problem C5: ***Corresponding Problem of Theorem C5.***

The inter-temporal Roy's identity under stochastic preferences is derived from the consumer problem in which the preference or utility function of the consumer in period 1 is known to be $u^{1(1)}(x_1)$. His future preferences are not known with certainty. In particular, his utility function in period $k \in \{2,3,\cdots,T\}$ is known to be $u^{k(\upsilon_k)}(x_k)$ with probability $\rho_k^{\upsilon_k}$ for $\upsilon_k \in \{1,2,\cdots,\overline{m}_k\}$. We use $\tilde{\upsilon}_k$ to denote the random variable with range $\upsilon_k \in \{1,2,\cdots,\overline{m}_k\}$ and corresponding probabilities $\{\rho_k^1, \rho_k^2, \cdots, \rho_k^{\overline{m}_k}\}$. The discount factor is embodied in the utility function.

The consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ \sum_{k=1}^{T} \sum_{\upsilon_k=1}^{\overline{m}_k} \rho_k^{\upsilon_k} \delta_1^k u^{k(\upsilon_k)}(x_k) \right\}$$

$$= E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ u^{1(1)}(x_1) + \sum_{k=2}^{T} \sum_{\upsilon_k=1}^{\overline{m}_k} \rho_k^{\upsilon_k} \delta_1^k u^{k(\upsilon_k)}(x_k) \right\}$$

subject to the budget constraint characterized by the wealth dynamic

$$W_{k+1} = W_k - \sum_{h=1}^{n_k} p_k^h x_k^h + r(W_k - \sum_{h=1}^{n_k} p_k^h x_k^h) + Y_{k+1}, \qquad W_1 = W_1^0.$$

C6. Inter-temporal Roy's Identity under Stochastic Life-span and Preferences

$$\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0, p)}{\partial p_\ell^j} \div \frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0, p)}{\partial W_\ell^0} \equiv -\varphi_\ell^{(\upsilon_\ell)j}(W_\ell^0, p), \text{ for } j \in \{1,2,\cdots,n_\ell\},$$

$$\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0, p)}{\partial p_h^k} \div \frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0, p)}{\partial W_\ell^0} \equiv -\sum_{\upsilon_{\ell+1}=1}^{\overline{m}_{\ell+1}} \rho_{\ell+1}^{\upsilon_{\ell+1}} \sum_{\upsilon_{\ell+2}=1}^{\overline{m}_{\ell+2}} \rho_{\ell+2}^{\upsilon_{\ell+2}} \cdots$$

$$\cdots \sum_{\upsilon_h=1}^{\overline{m}_h} \rho_h^{\upsilon_h} \; \delta_{\ell+1}^{h} \frac{\partial v^{h(\upsilon_h)}(W_h^{\upsilon_\ell \upsilon_{\ell+1} \cdots \upsilon_{h-1}}, p)}{\partial W_h^{\upsilon_\ell \upsilon_{\ell+1} \cdots \upsilon_{h-1}}} \varphi_h^{(\upsilon_h)k} (W_h^{\upsilon_\ell \upsilon_{\ell+1} \cdots \upsilon_{h-1}}, p) \; (1+r)^{-(h-\ell)}$$

$$\div \left[\sum_{\varpi_{\ell+1}=1}^{\overline{m}_{\ell+1}} \rho_{\ell+1}^{\varpi_{\ell+1}} \sum_{\varpi_{\ell+2}=1}^{\overline{m}_{\ell+2}} \rho_{\ell+2}^{\varpi_{\ell+2}} \cdots \sum_{\varpi_h=1}^{\overline{m}_h} \rho_h^{\varpi_h} \; \delta_{\ell+1}^{h} \frac{\partial v^{h(\varpi_h)}(W_h^{\varpi_\ell \varpi_{\ell+1} \cdots \varpi_{h-1}}, p)}{\partial W_h^{\varpi_\ell \varpi_{\ell+1} \cdots \varpi_{h-1}}} \right];$$

or in an alternative form

$$\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0, p)}{\partial W_\ell^0} \equiv -\frac{\sum_{\zeta=h}^{T} \gamma_\zeta}{\sum_{\zeta=\ell}^{T} \gamma_\zeta} \sum_{\upsilon_{\ell+1}=1}^{\overline{m}_{\ell+1}} \rho_{\ell+1}^{\upsilon_{\ell+1}} \sum_{\upsilon_{\ell+2}=1}^{\overline{m}_{\ell+2}} \rho_{\ell+2}^{\upsilon_{\ell+2}} \cdots$$

$$\cdots \sum_{\upsilon_{h+1}=1}^{\overline{m}_{h+1}} \rho_h^{\upsilon_h} \; \delta_{\ell+1}^{h+1} \frac{\partial v^{h(\upsilon_h)}(W_h^{\upsilon_\ell \upsilon_{\ell+1} \cdots \upsilon_{h-1}}, p)}{\partial W_h^{\upsilon_\ell \upsilon_{\ell+1} \cdots \upsilon_{h-1}}} \; (1+r)^{h-\ell};$$

for $\ell \in \{1,2,\cdots,T\}$, $h \in \{\ell+1, \ell+2, \cdots, T\}$, $k \in \{1,2,\cdots,n_h\}$ and $\upsilon_\ell \in \{1,2,\cdots,\overline{m}_\ell\}$, where

$W_\ell = W_\ell^0$,

$W_{\ell+1}^{\upsilon_\ell} = (1+r)[W_\ell^0 - p_\ell \varphi^{(\upsilon_\ell)}{}_\ell(W_\ell^0, p)] + Y_{\ell+1}$,

$W_{\ell+2}^{\upsilon_\ell \upsilon_{\ell+1}} = (1+r)[W_{\ell+1}^{\upsilon_\ell} - p_{\ell+1}\varphi_{\ell+1}^{(\upsilon_{\ell+1})}(W_{\ell+1}^{\upsilon_\ell}, p)] + Y_{\ell+2}$,

if preference is $u^{\ell+1(\upsilon_{\ell+1})}(x_{\ell+1})$ in period $\ell+1$;

$W_{\ell+3}^{\upsilon_\ell \upsilon_{\ell+1} \upsilon_{\ell+2}} = (1+r)[W_{\ell+2}^{\upsilon_\ell \upsilon_{\ell+1}} - p_{\ell+2}\varphi_{\ell+2}^{(\upsilon_{\ell+1})}(W_{\ell+2}^{\upsilon_\ell \upsilon_{\ell+1}}, p)] + Y_{\ell+3}$,

if preference is $u^{\ell+2(\upsilon_{\ell+2})}(x_{\ell+2})$ in period $\ell+2$;

$\vdots \qquad \vdots$

$W_T^{\upsilon_\ell, \upsilon_{\ell+1} \cdots \upsilon_{T-1}} = (1+r)[W_{T-1}^{\upsilon_\ell \upsilon_{\ell+1} \cdots \upsilon_{T-2}} - p_{T-1}\varphi_{T-1}^{(\upsilon_{T-1})}(W_{T-1}^{\upsilon_\ell \upsilon_{\ell+1} \cdots \upsilon_{T-2}})] + Y_T$;

if preference is $u^{T-1(\upsilon_{\ell+2})}(x_{T-1})$ in period $T-1$;

$W_{T+1}^{\upsilon_\ell, \upsilon_{\ell+1} \cdots \upsilon_T} = (1+r)[W_T^{\upsilon_\ell \upsilon_{\ell+1} \cdots \upsilon_{T-1}} - p_T \varphi_T^{(\upsilon\ )}(W_T^{\upsilon_\ell \upsilon_{\ell+1} \cdots \upsilon_{T-1}})] + Y_{T+1} = 0$. ■

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*
D.W.K. Yeung: Random Horizon Stocahstic dynamic Slutsky Equation under Preference Uncertainty, Applied Mathematical Sciences, Vol. 8, 2014, pp.7311-7340.

Problem C6: ***Corresponding Problem of Theorem C6.***

The inter-temporal Roy's identity under stochastic life-span and preferences is derived from the consumer problem in which the consumer's life-span involves $\hat{T}$ periods where $\hat{T}$ is a random variable with range $\{1,2,\cdots,T\}$ and corresponding probabilities $\{\gamma_1, \gamma_2, \cdots, \gamma_T\}$. Conditional upon the reaching of period τ, the probability of the consumer's life-span would last up to periods $\tau, \tau+1, \cdots, T$ becomes respectively:

$$\frac{\gamma_\tau}{\sum_{\zeta=\tau}^{T} \gamma_\zeta}, \frac{\gamma_{\tau+1}}{\sum_{\zeta=\tau}^{T} \gamma_\zeta}, \cdots, \frac{\gamma_T}{\sum_{\zeta=\tau}^{T} \gamma_\zeta}.$$

The preference or utility function of the consumer in period 1 is known to be $u^{1(1)}(x_1)$. His future preferences are not known with certainty. In particular, his utility function in period $k \in \{2,3,\cdots,T\}$ is known to be $u^{k(\upsilon_k)}(x_k)$ with probability $\rho_k^{\upsilon_k}$ for $\upsilon_k \in \{1,2,\cdots,\overline{m}_k\}$ if he survives in period k. We use $\tilde{\upsilon}_k$ to denote the random variable with range $\upsilon_k \in \{1,2,\cdots,\overline{m}_k\}$ and corresponding probabilities $\{\rho_k^1, \rho_k^2, \cdots, \rho_k^{\overline{m}_k}\}$. The discount factor is embodied in the utility function.

The consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ \sum_{\hat{T}=1}^{T}\gamma_{\hat{T}} \sum_{k=1}^{\hat{T}} \sum_{\upsilon_k=1}^{\overline{m}_k}\rho_k^{\upsilon_k}\delta_1^k u^{k(\upsilon_k)}(x_k) \right\}$$

$$= E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ u^{1(1)}(x_1) + \sum_{\hat{T}=2}^{T}\gamma_{\hat{T}} \sum_{k=2}^{\hat{T}} \sum_{\upsilon_k=1}^{\overline{m}_k}\rho_k^{\upsilon_k}\delta_1^k u^{k(\upsilon_k)}(x_k) \right\}$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1} = W_k - \sum_{h=1}^{n_k} p_k^h x_k^h + r(W_k - \sum_{h=1}^{n_k} p_k^h x_k^h) + Y_{k+1}, \qquad W_1 = W_1^0.$$

C7. Inter-temporal Roy's Identity under Stochastic Income and Preferences

$$\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0, p)}{\partial p_\ell^j} \div \frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0, p)}{\partial W_\ell^0} \equiv -\varphi_\ell^{(\upsilon_\ell)j}(W_\ell^0, p), \text{ for } j \in \{1,2,\cdots,n_\ell\};$$

$$\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0, p)}{\partial p_h^k} \div \frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0, p)}{\partial W_\ell^0} \equiv -\sum_{j_{\ell+1}=1}^{m_{\ell+1}}\lambda_{\ell+1}^{j_{\ell+1}}\sum_{j_{\ell+2}=1}^{m_{\ell+2}}\lambda_{\ell+2}^{j_{\ell+2}}\cdots\sum_{j_h=1}^{m_h}\lambda_h^{j_h}\sum_{\upsilon_{\ell+1}=1}^{\overline{m}_{\ell+1}}\rho_{\ell+1}^{\upsilon_{\ell+1}}\sum_{\upsilon_{\ell+2}=1}^{\overline{m}_{\ell+2}}\rho_{\ell+2}^{\upsilon_{\ell+2}}\cdots$$

$$\cdots\sum_{\upsilon_h=1}^{\overline{m}_h}\rho_h^{\upsilon_h}\ \delta_{\ell+1}^h \frac{\partial v^{h(\upsilon_h)}(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}}, p)}{\partial W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}}}$$

$$\times\ \varphi_h^{(\upsilon_h)k}(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}}, p)\ (1+r)^{-(h-\ell)}$$

$$\div\left[\sum_{w_{\ell+1}=1}^{m_{\ell+1}}\lambda_{\ell+1}^{w_{\ell+1}}\sum_{w_{\ell+2}=1}^{m_{\ell+2}}\lambda_{\ell+2}^{w_{\ell+2}}\cdots\sum_{w_h=1}^{m_h}\lambda_h^{w_h}\sum_{\varpi_{\ell+1}=1}^{\overline{m}_{\ell+1}}\rho_{\ell+1}^{\varpi_{\ell+1}}\sum_{\varpi_{\ell+2}=1}^{\overline{m}_{\ell+2}}\rho_{\ell+2}^{\varpi_{\ell+2}}\cdots\sum_{\varpi_h=1}^{\overline{m}_h}\rho_h^{\varpi_h}\right.$$

$$\left.\times\ \delta_{\ell+1}^h \frac{\partial v^{h(\varpi_h)}(W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_h^{w_h};\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}}, p)}{\partial W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_h^{w_h};\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}}}\right];$$

or in an alternative form:

$$\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0, p)}{\partial p_h^k} \equiv -\sum_{j_{\ell+1}=1}^{m_{\ell+1}}\lambda_{\ell+1}^{j_{\ell+1}}\sum_{j_{\ell+2}=1}^{m_{\ell+2}}\lambda_{\ell+2}^{j_{\ell+2}}\cdots\sum_{j_h=1}^{m_h}\lambda_h^{j_h}\sum_{\upsilon_{\ell+1}=1}^{\overline{m}_{\ell+1}}\rho_{\ell+1}^{\upsilon_{\ell+1}}\sum_{\upsilon_{\ell+2}=1}^{\overline{m}_{\ell+2}}\rho_{\ell+2}^{\upsilon_{\ell+2}}\cdots\sum_{\upsilon_h=1}^{\overline{m}_h}\rho_h^{\upsilon_h}$$

$$\delta_{\ell+1}^h \frac{\partial v^{h(\upsilon_h)}(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}}, p)}{\partial W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}}}\ \varphi_h^{(\upsilon_h)k}(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}}, p);$$

for $\ell \in \{1,2,\cdots,T\}$, $h \in \{\ell+1, \ell+2, \cdots, T\}$, $k \in \{1,2,\cdots,n_h\}$ and $\upsilon_\ell \in \{1,2,\cdots,\overline{m}_\ell\}$, where

$W_\ell = W_\ell^0$,

$W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}};\upsilon_\ell} = (1+r)[W_\ell^0 - p_\ell\varphi_\ell^{(\upsilon_\ell)}(W_\ell, p)] + \theta_{\ell+1}^{j_{\ell+1}}$,

$W_{\ell+2}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}};\upsilon_\ell\upsilon_{\ell+1}} = (1+r)[W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}};\upsilon_\ell} - p_{\ell+1}\varphi_{\ell+1}^{(\upsilon_{\ell+1})}(W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}};\upsilon_\ell}, p)] + \theta_{\ell+2}^{j_{\ell+2}}$,

if preference is $u^{\ell+1(\upsilon_{\ell+1})}(x_{\ell+1})$ in period $\ell+1$;

$$W_{\ell+3}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\theta_{\ell+3}^{j_{\ell+3}};\upsilon_\ell\upsilon_{\ell+1}\upsilon_{\ell+2}} = (1+r)[W_{\ell+2}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}};\upsilon_\ell\upsilon_{\ell+1}} - p_{\ell+2}\varphi_{\ell+2}^{(\upsilon_{\ell+1})}(W_{\ell+2}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}};\upsilon_\ell\upsilon_{\ell+1}}, p)] + \theta_{\ell+3}^{j_{\ell+3}},$$

if preference is $u^{\ell+2(\upsilon_{\ell+2})}(x_{\ell+2})$ in period $\ell+2$;

$\vdots$ $\vdots$

$$W_{T}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T}^{j_T};\upsilon_\ell,\upsilon_{\ell+1}\cdots\upsilon_{T-1}}$$

$$= (1+r)[W_{T-1}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T-1}^{j_{T-1}};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{T-2}} - p_{T-1}\varphi_{T-1}^{(\upsilon_{T-1})}(W_{T-1}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T-1}^{j_{T-1}};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{T-2}})] + \theta_T^{j_T};$$

if preference is $u^{T-1(\upsilon_{\ell+2})}(x_{T-1})$ in period $T-1$;

$$W_{T+1}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T+1}^{j_{T+1}};\upsilon_\ell,\upsilon_{\ell+1}\cdots\upsilon_{T}}$$

$$= (1+r)[W_{T}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T}^{j_T};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{T-1}} - p_T\varphi_T^{(\upsilon\ )}(W_{T}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T}^{j_T};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{T-1}})] + \theta_{T+1}^{j_{T+1}} = 0. \quad \blacksquare$$

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*
D.W.K. Yeung: Random Horizon Stocahstic dynamic Slutsky Equation under Preference Uncertainty, Applied Mathematical Sciences, Vol. 8, 2014, pp.7311-7340.

Problem C7: ***Corresponding Problem of Theorem C7.***

The inter-temporal Roy's identity under stochastic income and preferences is derived from the consumer problem in which the preference or utility function of the consumer in period 1 is known to be $u^{1(1)}(x_1)$. His future preferences are not known with certainty. In particular, his utility function in period $k \in \{2,3,\cdots,T\}$ is known to be $u^{k(\upsilon_k)}(x_k)$ with probability $\rho_k^{\upsilon_k}$ for $\upsilon_k \in \{1,2,\cdots,\overline{m}_k\}$. We use $\tilde{\upsilon}_k$ to denote the random variable with range $\upsilon_k \in \{1,2,\cdots,\overline{m}_k\}$ and corresponding probabilities $\{\rho_k^1,\rho_k^2,\cdots,\rho_k^{\overline{m}_k}\}$. The discount factor is embodied in the utility function.

The consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ \sum_{k=1}^{T}\sum_{\upsilon_k=1}^{\overline{m}_k}\rho_k^{\upsilon_k}\delta_1^k u^{k(\upsilon_k)}(x_k) \right\}$$

$$= E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ u^{1(1)}(x_1) + \sum_{k=2}^{T}\sum_{\upsilon_k=1}^{\overline{m}_k}\rho_k^{\upsilon_k}\delta_1^k u^{k(\upsilon_k)}(x_k) \right\}$$

subject to the budget constraint characterized by the wealth dynamic

$$W_{k+1} = W_k - \sum_{h=1}^{n_k} p_k^h x_k^h + r(W_k - \sum_{h=1}^{n_k} p_k^h x_k^h) + \theta_{k+1}, \qquad W_1 = W_1^0.$$

where

θ_k is the random income that the consumer will receive in period k; and θ_k, for $k \in \{2,\cdots,T\}$, is a set of statistically independent random variables, and $E_{\theta_1,\theta_2,\cdots,\theta_T}$ is the expectation operation with respect to the statistics of $\theta_2,\theta_3,\cdots,\theta_T$. The random variable θ_k has a non-negative range $\{\theta_k^1,\theta_k^2,\cdots,\theta_k^{m_k}\}$ with corresponding probabilities $\{\lambda_k^1,\lambda_k^2,\cdots,\lambda_k^{m_k}\}$, for $k \in \{2,\cdots,T\}$.

C8. Inter-temporal Roy's Identity under Stochastic Income, Life-span and Preferences

$$\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial p_\ell^j} \div \frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial W_\ell^0} \equiv -\varphi_\ell^{(\upsilon_\ell)j}(W_\ell^0,p), \text{ for } j\in\{1,2,\cdots,n_\ell\};$$

$$\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial p_h^k} \div \frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial W_\ell^0} \equiv -\sum_{j_{\ell+1}=1}^{m_{\ell+1}}\lambda_{\ell+1}^{j_{\ell+1}}\sum_{j_{\ell+2}=1}^{m_{\ell+2}}\lambda_{\ell+2}^{j_{\ell+2}}\cdots\sum_{j_h=1}^{m_h}\lambda_h^{j_h}\sum_{\upsilon_{\ell+1}=1}^{\overline{m}_{\ell+1}}\rho_{\ell+1}^{\upsilon_{\ell+1}}\sum_{\upsilon_{\ell+2}=1}^{\overline{m}_{\ell+2}}\rho_{\ell+2}^{\upsilon_{\ell+2}}\cdots$$

$$\cdots\sum_{\upsilon_h=1}^{\overline{m}_h}\rho_h^{\upsilon_h}\quad\delta_{\ell+1}^h\quad\frac{\partial v^{h(\upsilon_h)}(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}},p)}{\partial W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}}}\quad\varphi_h^{(\upsilon_h)k}(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}},p)$$

$$(1+r)^{-(h-\ell)}$$

$$\div\Bigg[\sum_{w_{\ell+1}=1}^{m_{\ell+1}}\lambda_{\ell+1}^{w_{\ell+1}}\sum_{w_{\ell+2}=1}^{m_{\ell+2}}\lambda_{\ell+2}^{w_{\ell+2}}\cdots\sum_{w_h=1}^{m_h}\lambda_h^{w_h}\sum_{\varpi_{\ell+1}=1}^{\overline{m}_{\ell+1}}\rho_{\ell+1}^{\varpi_{\ell+1}}\sum_{\varpi_{\ell+2}=1}^{\overline{m}_{\ell+2}}\rho_{\ell+2}^{\varpi_{\ell+2}}\cdots\sum_{\varpi_h=1}^{\overline{m}_h}\rho_h^{\varpi_h}$$

$$\times\,\delta_{\ell+1}^h\frac{\partial v^{h(\varpi_h)}(W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_h^{w_h};\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}},p)}{\partial W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_h^{w_h};\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}}}\Bigg];$$

or in an alternative form:

$$\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial p_h^k} \equiv -\frac{\sum_{\zeta=h}^{T}\gamma_\zeta}{\sum_{\zeta=\ell}^{T}\gamma_\zeta}\sum_{j_{\ell+1}=1}^{m_{\ell+1}}\lambda_{\ell+1}^{j_{\ell+1}}\sum_{j_{\ell+2}=1}^{m_{\ell+2}}\lambda_{\ell+2}^{j_{\ell+2}}\cdots\sum_{j_h=1}^{m_h}\lambda_h^{j_h}\sum_{\upsilon_{\ell+1}=1}^{\overline{m}_{\ell+1}}\rho_{\ell+1}^{\upsilon_{\ell+1}}\sum_{\upsilon_{\ell+2}=1}^{\overline{m}_{\ell+2}}\rho_{\ell+2}^{\upsilon_{\ell+2}}\cdots\sum_{\upsilon_h=1}^{\overline{m}_h}\rho_h^{\upsilon_h}$$

$$\delta_{\ell+1}^h\frac{\partial v^{h(\upsilon_h)}(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}},p)}{\partial W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}}}\varphi_h^{(\upsilon_h)k}(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}},p);$$

for $\ell\in\{1,2,\cdots,T\}$, $h\in\{\ell+1,\ell+2,\cdots,T\}$, $k\in\{1,2,\cdots,n_h\}$ and $\upsilon_\ell\in\{1,2,\cdots,\overline{m}_\ell\}$,
where

$W_\ell = W_\ell^0$,

$W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}};\upsilon_\ell} = (1+r)[W_\ell^0 - p_\ell\varphi^{(\upsilon_\ell)}{}_\ell(W_\ell^0,p)] + \theta_{\ell+1}^{j_{\ell+1}}$,

$W_{\ell+2}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}};\upsilon_\ell\upsilon_{\ell+1}} = (1+r)[W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}};\upsilon_\ell} - p_{\ell+1}\varphi_{\ell+1}^{(\upsilon_{\ell+1})}(W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}};\upsilon_\ell},p)] + \theta_{\ell+2}^{j_{\ell+2}}$,

if preference is $u^{\ell+1(\upsilon_{\ell+1})}(x_{\ell+1})$ in period $\ell+1$;

$W_{\ell+3}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\theta_{\ell+3}^{j_{\ell+3}};\upsilon_\ell\upsilon_{\ell+1}\upsilon_{\ell+2}} = (1+r)[W_{\ell+2}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}};\upsilon_\ell\upsilon_{\ell+1}} - p_{\ell+2}\varphi_{\ell+2}^{(\upsilon_{\ell+1})}(W_{\ell+2}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}};\upsilon_\ell\upsilon_{\ell+1}},p)] + \theta_{\ell+3}^{j_{\ell+3}}$,

if preference is $u^{\ell+2(\upsilon_{\ell+2})}(x_{\ell+2})$ in period $\ell+2$;

$\vdots \qquad\qquad \vdots$

$W_T^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_T^{jT};\upsilon_\ell,\upsilon_{\ell+1}\cdots\upsilon_{T-1}}$

$= (1+r)[W_{T-1}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T-1}^{jT-1};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{T-2}} - p_{T-1}\varphi_{T-1}^{(\upsilon_{T-1})}(W_{T-1}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T-1}^{jT-1};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{T-2}})] + \theta_T^{j_T}$;

if preference is $u^{T-1(\upsilon_{\ell+2})}(x_{T-1})$ in period $T-1$;

$W_{T+1}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T+1}^{jT+1};\upsilon_\ell,\upsilon_{\ell+1}\cdots\upsilon_T}$

$= (1+r)[W_T^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_T^{jT};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{T-1}} - p_T\varphi_T^{(\upsilon\ )}(W_T^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_T^{jT};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{T-1}})] + \theta_{T+1}^{j_{T+1}} = 0$. ■

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*

D.W.K. Yeung: Random Horizon Stocahstic dynamic Slutsky Equation under Preference Uncertainty, Applied Mathematical Sciences, Vol. 8, 2014, pp.7311-7340.

Problem C8: ***Corresponding Problem of Theorem C8.***

The inter-temporal Roy's identity under stochastic income, life-span and preferences is derived from the consumer problem in which the consumer's life-span involves $\hat{T}$ periods where $\hat{T}$ is a random variable with range $\{1,2,\cdots,T\}$ and corresponding probabilities $\{\gamma_1,\gamma_2,\cdots,\gamma_T\}$. Conditional upon the reaching of period τ, the probability of the consumer's life-span would last up to periods $\tau,\tau+1,\cdots,T$ becomes respectively:

$$\frac{\gamma_\tau}{\sum_{\zeta=\tau}^{T}\gamma_\zeta},\frac{\gamma_{\tau+1}}{\sum_{\zeta=\tau}^{T}\gamma_\zeta},\cdots,\frac{\gamma_T}{\sum_{\zeta=\tau}^{T}\gamma_\zeta}.$$

The preference or utility function of the consumer in period 1 is known to be $u^{1(1)}(x_1)$. His future preferences are not known with certainty. In particular, his utility function in period $k\in\{2,3,\cdots,T\}$ is known to be $u^{k(\upsilon_k)}(x_k)$ with probability $\rho_k^{\upsilon_k}$ for $\upsilon_k\in\{1,2,\cdots,\overline{m}_k\}$ if he survives in period k. We use $\tilde{\upsilon}_k$ to denote the random variable with range $\upsilon_k\in\{1,2,\cdots,\overline{m}_k\}$ and corresponding probabilities $\{\rho_k^1,\rho_k^2,\cdots,\rho_k^{\overline{m}_k}\}$. The discount factor is embodied in the utility function.

The consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{\sum_{\hat{T}=1}^{T}\gamma_{\hat{T}}\sum_{k=1}^{\hat{T}}\sum_{\upsilon_k=1}^{\overline{m}_k}\rho_k^{\upsilon_k}\delta_1^k u^{k(\upsilon_k)}(x_k)\right\}$$

$$=E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{u^{1(1)}(x_1)+\sum_{\hat{T}=2}^{T}\gamma_{\hat{T}}\sum_{k=2}^{\hat{T}}\sum_{\upsilon_k=1}^{\overline{m}_k}\rho_k^{\upsilon_k}\delta_1^k u^{k(\upsilon_k)}(x_k)\right\},$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1}=W_k-\sum_{h=1}^{n_k}p_k^hx_k^h+r(W_k-\sum_{h=1}^{n_k}p_k^hx_k^h)+\theta_{k+1},\qquad W_1=W_1^0.$$

where

θ_k is the random income that the consumer will receive in period k; and θ_k, for $k\in\{2,\cdots,T\}$, is a set of statistically independent random variables, and $E_{\theta_1,\theta_2,\cdots,\theta_T}$ is the expectation operation with respect to the statistics of $\theta_2,\theta_3,\cdots,\theta_T$. The random variable θ_k has a non-negative range $\{\theta_k^1,\theta_k^2,\cdots,\theta_k^{m_k}\}$ with corresponding probabilities $\{\lambda_k^1,\lambda_k^2,\cdots,\lambda_k^{m_k}\}$, for $k\in\{2,\cdots,T\}$.

C9. Dynamic Slutsky Equation

$$\frac{\partial\varphi_\ell^h(W_\ell^0,p_\ell,p_{\ell+1},\cdots,p_T)}{\partial p_k^j}=\frac{\partial\psi_\ell^h(\hat{v}_\ell^{W_\ell^0},p_\ell,p_{\ell+1},\cdots,p_T)}{\partial p_k^j}$$

$$-\frac{\partial\varphi_\ell^h(W_\ell^0,p_\ell,p_{\ell+1},\cdots,p_T)}{\partial W_\ell^0}\,\varphi_k^j(W_k^0,p_k,p_{k+1},\cdots,p_T)\,(1+r)^{-(k-\ell)},$$

for $j\in\{1,2,\cdots,n_k\}$ and $k\in\{\ell,\ell+1,\cdots,T\}$. ■

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*

D.W.K. Yeung: Optimal Consumption under an Uncertain Inter-temporal Budget: Stochastic Dynamic Slutsky Equations, Vestnik St Petersburg University: Mathematics (Springer), Vol. 10, 2013, pp.121-141.

Problem C9: ***Corresponding Problem of Theorem C9.***

The dynamic Slutsky equation is derived from the consumer problem in which the consumer maximizes his inter-temporal utility

$$u^1(x_1^1, x_1^2, \cdots, x_1^{n_1}) + \sum_{k=2}^{T} \delta_2^k u^k(x_k^1, x_k^2, \cdots, x_k^{n_k})$$

$$= u^1(x_1) + \sum_{k=2}^{T} \delta_2^k u^k(x_k) = \sum_{k=1}^{T} \delta_1^k u^k(x_k)$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1} = W_k - \sum_{h=1}^{\eta_k} p_k^h x_k^h + r(W_k - \sum_{h=1}^{\eta_k} p_k^h x_k^h) + Y_{k+1}, \qquad W_1 = W_1^0,$$

where

$x_k = (x_k^1, x_k^2, \cdots, x_k^{n_k})$ is the vector of quantities of goods consumed in period k, $p_k = (p_k^1, p_k^2, \cdots, p_k^{n_k})$ is price vector, r is the interest rate, Y_k is the income that the consumer will receive in period k, $\delta_2^k = \left(\prod_{c=2}^{k} \beta_c \right)$ is the discount factor with β_τ being the consumer's subjective one-period discount factor for the duration from period $\tau - 1$ to period τ, $\beta_1 = 1$ for the discount factor in the initial period 1 and $\delta_1^k = \left(\prod_{c=1}^{k} \beta_c \right) = \delta_2^k = \left(\prod_{c=2}^{k} \beta_c \right)$. The period k utility function $u^k(x_k^1, x_k^2, \cdots, x_k^{n_k})$ is continuously differentiable and quasi-concave yielding convex level (indifference) curves. The time preference factor is embodied in the utility function. The time preference factor is embodied in the utility function. The amount of unconsumed wealth $W_k - p_k x_k$ in period k will generate an interest income $r(W_k - p_k x_k)$ in period $k+1$.

In addition $\varphi_\ell^h(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)$ is the ordinary demand function of commodity h in period ℓ, and $\psi_\ell^h(\hat{v}_\ell^{W_\ell^0}, p_\ell, p_{\ell+1}, \cdots, p_T)$ is the wealth compensated demand function of commodity h in period ℓ.

C10. Dynamic Slutsky Equation under Stochastic Income

$$\frac{\partial \varphi_\ell^h(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_\ell^j} = \frac{\partial \psi_\ell^h(\hat{v}_\ell^{W_\ell^0}, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_\ell^j}$$

$$- \frac{\partial \varphi_\ell^h(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial W_\ell^0} \varphi_\ell^j(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T),$$

for $j \in \{1, 2, \cdots, n_\ell\}$,

and

$$\frac{\partial \varphi_\ell^h(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_k^j} = \frac{\partial \psi_\ell^h(\hat{v}_\ell^{W_\ell^0}, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_k^j}$$

$$-\frac{\partial \varphi_\ell^h(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial W_\ell^0} \sum_{j_{\ell+1}=1}^{m_{\ell+1}} \lambda_{\ell+1}^{j_{\ell+1}} \sum_{j_{\ell+2}=1}^{m_{\ell+2}} \lambda_{\ell+2}^{j_{\ell+2}} \cdots \sum_{j_k=1}^{m_k} \lambda_k^{j_k}$$

$$\times \frac{\delta_{\ell+1}^k \dfrac{\partial v^k(W_k^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_k^{j_k}}, p)}{\partial W_k^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_k^{j_k}}}}{\sum_{w_{\ell+1}=1}^{m_{\ell+1}} \lambda_{\ell+1}^{w_{\ell+1}} \sum_{w_{\ell+2}=1}^{m_{\ell+2}} \lambda_{\ell+2}^{w_{\ell+2}} \cdots \sum_{w_k=1}^{m_k} \lambda_k^{w_k} \delta_{\ell+1}^k \dfrac{\partial v^k(W_k^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_k^{w_k}}, p)}{\partial W_k^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_k^{w_k}}}} \varphi_k^j(W_k^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_k^{j_k}}, p)\,(1+r)^{-(k-\ell)},$$

for $\ell \in \{1,2,\cdots,T-1\}$, $k \in \{k+1, k+2, \cdots, T\}$ and $j \in \{1,2,\cdots,n_k\}$. ■

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*
D.W.K. Yeung: Optimal Consumption under an Uncertain Inter-temporal Budget: Stochastic Dynamic Slutsky Equations, Vestnik St Petersburg University: Mathematics (Springer), Vol. 10, 2013, pp.121-141.

Problem C10: ***Corresponding Problem of Theorem C10.***
The dynamic Slutsky equation under stochastic income is derived from the consumer problem in which the consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3\cdots,\theta_T}\left\{ \sum_{k=1}^{T} \delta_1^k u^k(x_k^1, x_k^2, \cdots, x_k^{n_k}) \right\} = E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ \sum_{k=1}^{T} \delta_1^k u^k(x_k) \right\}$$

subject to the budget constraint characterized by the stochastic wealth dynamics

$$W_{k+1} = (1+r)(W_k - p_k x_k) + \theta_{k+1}, \qquad W_1 = W_1^0,$$

where
θ_k is the random income that the consumer will receive in period k; and θ_k, for $k \in \{2,\cdots,T\}$, is a set of statistically independent random variables, and $E_{\theta_1,\theta_2,\cdots,\theta_T}$ is the expectation operation with respect to the statistics of $\theta_2, \theta_3, \cdots, \theta_T$. The random variable θ_k has a non-negative range $\{\theta_k^1, \theta_k^2, \cdots, \theta_k^{m_k}\}$ with corresponding probabilities $\{\lambda_k^1, \lambda_k^2, \cdots, \lambda_k^{m_k}\}$, for $k \in \{2,\cdots,T\}$.

C11. Dynamic Slutsky Equation under Stochastic Life-span

$$\frac{\partial \varphi_\ell^h(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_k^j} = \frac{\partial \psi_\ell^h(\hat{v}_\ell^{W_\ell^0}, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_k^j}$$

$$-\frac{\partial \varphi_\ell^h(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial W_\ell^0} (1+r)^{-(k-\ell)} \varphi_k^j(W_k^0, p_k, p_{k+1}, \cdots, p_T),$$

for $j \in \{1,2,\cdots,n_k\}$ and $k \in \{\ell, \ell+1, \cdots, T\}$. ■

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*
D.W.K. Yeung: Optimal Consumption under Uncertainties: Random Horizon Stochastic Dynamic Roy's Identity and Slutsky Equation, Applied Mathematics, Vol.5,

2014, pp.263-284.

Problem C11: ***Corresponding Problem of Theorem C11.***
The dynamic Slutsky equation under stochastic life-span is derived from the consumer problem in which the consumer's life-span involves $\hat{T}$ periods where $\hat{T}$ is a random variable with range $\{1,2,\cdots,T\}$ and corresponding probabilities $\{\gamma_1,\gamma_2,\cdots,\gamma_T\}$. Conditional upon the reaching of period τ, the probability of the consumer's life-span would last up to periods $\tau,\tau+1,\cdots,T$ becomes respectively

$$\frac{\gamma_\tau}{\sum_{\zeta=\tau}^{T}\gamma_\zeta},\ \frac{\gamma_{\tau+1}}{\sum_{\zeta=\tau}^{T}\gamma_\zeta},\cdots,\frac{\gamma_T}{\sum_{\zeta=\tau}^{T}\gamma_\zeta}.$$

The consumer maximizes his expected inter-temporal utility

$$\sum_{\hat{T}=1}^{T}\gamma_{\hat{T}}\sum_{k=1}^{\hat{T}}\delta_1^k u^k(x_k),$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1}=W_k-\sum_{h=1}^{n_k}p_k^h x_k^h+r(W_k-\sum_{h=1}^{n_k}p_k^h x_k^h)+Y_{k+1},\qquad W_1=W_1^0.$$

where

r is the interest rate, Y_k is the income that the consumer will receive in period k.

C12. Dynamic Slutsky Equation under Stochastic Income and Life-span

$$\frac{\partial\varphi_\ell^h(W_\ell^0,p_\ell,p_{\ell+1},\cdots,p_T)}{\partial p_\ell^j}=\frac{\partial\psi_\ell^h(\hat{v}_\ell^{W_\ell^0},p_\ell,p_{\ell+1},\cdots,p_T)}{\partial p_\ell^j}-\frac{\partial\varphi_\ell^h(W_\ell^0,p_\ell,p_{\ell+1},\cdots,p_T)}{\partial W_\ell^0}\varphi_\ell^j(W_\ell^0,p_\ell,p_{\ell+1},\cdots,p_T),$$

for $j\in\{1,2,\cdots,n_\ell\}$,

and

$$\frac{\partial\varphi_\ell^h(W_\ell^0,p_\ell,p_{\ell+1},\cdots,p_T)}{\partial p_k^j}=\frac{\partial\psi_\ell^h(\hat{v}_\ell^{W_\ell^0},p_\ell,p_{\ell+1},\cdots,p_T)}{\partial p_k^j}-\frac{\partial\varphi_\ell^h(W_\ell^0,p_\ell,p_{\ell+1},\cdots,p_T)}{\partial W_\ell^0}\sum_{j_{\ell+1}=1}^{m_{\ell+1}}\lambda_{\ell+1}^{j_{\ell+1}}\sum_{j_{\ell+2}=1}^{m_{\ell+2}}\lambda_{\ell+2}^{j_{\ell+2}}\cdots\sum_{j_k=1}^{m_k}\lambda_k^{j_k}$$

$$\times\frac{\delta_{\ell+1}^k\dfrac{\partial v^k(W_k^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_k^{j_k}},p)}{\partial W_k^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_k^{j_k}}}}{\sum_{w_{\ell+1}=1}^{m_{\ell+1}}\lambda_{\ell+1}^{w_{\ell+1}}\sum_{w_{\ell+2}=1}^{m_{\ell+2}}\lambda_{\ell+2}^{w_{\ell+2}}\cdots\sum_{w_k=1}^{m_k}\lambda_k^{w_k}\delta_{\ell+1}^k\dfrac{\partial v^k(W_k^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_k^{w_k}},p)}{\partial W_k^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_k^{w_k}}}}\varphi_\ell^j(W_k^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_k^{j_k}},p)(1+r)^{-(k-\ell)},$$

for $\ell\in\{1,2,\cdots,T-1\}$, $k\in\{k+1,k+2,\cdots,T\}$ and $j\in\{1,2,\cdots,n_k\}$. ■ ***References:***

D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.
D.W.K. Yeung: Optimal Consumption under Uncertainties: Random Horizon

Stochastic Dynamic Roy's Identity and Slutsky Equation, Applied Mathematics, Vol.5, 2014, pp.263-284.

Problem C12: ***Corresponding Problem of Theorem C12.***

The dynamic Slutsky equation under stochastic income and life-span is derived from the consumer problem in which the consumer's life-span involves $\hat{T}$ periods where $\hat{T}$ is a random variable with range $\{1,2,\cdots,T\}$ and corresponding probabilities $\{\gamma_1,\gamma_2,\cdots,\gamma_T\}$. Conditional upon the reaching of period τ, the probability of the consumer's life-span would last up to periods $\tau,\tau+1,\cdots,T$ becomes respectively

$$\frac{\gamma_\tau}{\sum_{\zeta=\tau}^{T}\gamma_\zeta},\frac{\gamma_{\tau+1}}{\sum_{\zeta=\tau}^{T}\gamma_\zeta},\cdots,\frac{\gamma_T}{\sum_{\zeta=\tau}^{T}\gamma_\zeta}.$$

The consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3,\cdots,\theta_{T+1}}\left\{\sum_{\hat{T}=1}^{T}\gamma_{\hat{T}}\sum_{k=1}^{\hat{T}}\delta_1^k u^k(x_k)\right\},$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1}=W_k-\sum_{h=1}^{n_k}p_k^h x_k^h+r(W_k-\sum_{h=1}^{n_k}p_k^h x_k^h)+\theta_{k+1}, \qquad W_1=W_1^0.$$

where

θ_k is the random income that the consumer will receive in period k; and θ_k, for $k\in\{2,\cdots,T\}$, is a set of statistically independent random variables, and $E_{\theta_1,\theta_2,\cdots,\theta_T}$ is the expectation operation with respect to the statistics of $\theta_2,\theta_3,\cdots,\theta_T$. The random variable θ_k has a non-negative range $\{\theta_k^1,\theta_k^2,\cdots,\theta_k^{m_k}\}$ with corresponding probabilities $\{\lambda_k^1,\lambda_k^2,\cdots,\lambda_k^{m_k}\}$, for $k\in\{2,\cdots,T\}$.

C13. Dynamic Slutsky Equation under Stochastic Preferences

$$\frac{\partial\varphi_\ell^{(\upsilon_\ell)h}(W_\ell^0,p)}{\partial p_\ell^{i_\ell}}=\frac{\partial\psi_\ell^{(\upsilon_\ell)h}(W_\ell^0,p)}{\partial p_\ell^{i_\ell}}-\frac{\partial\varphi_\ell^{(\upsilon_\ell)h}(W_\ell^0,p)}{\partial W_\ell^0}\,\varphi_\ell^{i_\ell}(W_\ell^0,p),$$

$$\frac{\partial\varphi_\ell^{(\upsilon_\ell)h}(W_\ell^0,p)}{\partial p_k^{i_k}}=\frac{\partial\psi_\ell^{(\upsilon_\ell)h}(W_\ell^0,p)}{\partial p_k^{i_k}}-\frac{\partial\varphi_\ell^{(\upsilon_\ell)h}(W_\ell^0,p)}{\partial W_\ell^0}\sum_{\upsilon_{\ell+1}=1}^{\overline{m}_{\ell+1}}\rho_{\ell+1}^{\upsilon_{\ell+1}}\sum_{\upsilon_{\ell+2}=1}^{\overline{m}_{\ell+2}}\rho_{\ell+2}^{\upsilon_{\ell+2}}\cdots$$

$$\cdots\sum_{\upsilon_k=1}^{\overline{m}_k}\rho_k^{\upsilon_k}\;\delta_{\ell+1}^k\frac{\partial v^{(\upsilon_k)k}(W_k^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{k-1}},p)}{\partial W_k^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{k-1}}}\;\varphi_k^{i_k}(W_k^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{k-1}},p)\,(1+r)^{-(\ell-k)}$$

$$\div\left[\sum_{\varpi_{\ell+1}=1}^{\overline{m}_{\ell+1}}\rho_{\ell+1}^{\varpi_{\ell+1}}\sum_{\varpi_{\ell+2}=1}^{\overline{m}_{\ell+2}}\rho_{\ell+2}^{\varpi_{\ell+2}}\cdots\sum_{\varpi_h=1}^{\overline{m}_h}\rho_h^{\varpi_h}\;\delta_{\ell+1}^h\frac{\partial v^{h(\varpi_h)}(W_h^{\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}},p)}{\partial W_h^{\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}}}\right],$$

for $\ell\in\{1,2,\cdots,T\}$, $k\in\{\ell+1,\ell+2,\cdots,T\}$, $i_k\in\{1,2,\cdots,n_k\}$, $h,i_\ell\in\{1,2,\cdots,n_\ell\}$ and $\upsilon_\ell\in\{1,2,\cdots,\overline{m}_\ell\}$. ■

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*
D.W.K. Yeung: Random Horizon Stocahstic dynamic Slutsky Equation under Preference Uncertainty, Applied Mathematical Sciences, Vol. 8, 2014, pp.7311-7340.

Problem C13: ***Corresponding Problem of Theorem C13.***

The dynamic Slutsky equation under stochastic preferences is derived from the consumer problem in which the preference or utility function of the consumer in period 1 is known to be $u^{1(1)}(x_1)$. His future preferences are not known with certainty. In particular, his utility function in period $k \in \{2,3,\cdots,T\}$ is known to be $u^{k(\upsilon_k)}(x_k)$ with probability $\rho_k^{\upsilon_k}$ for $\upsilon_k \in \{1,2,\cdots,\overline{m}_k\}$. We use $\tilde{\upsilon}_k$ to denote the random variable with range $\upsilon_k \in \{1,2,\cdots,\overline{m}_k\}$ and corresponding probabilities $\{\rho_k^1, \rho_k^2, \cdots, \rho_k^{\overline{m}_k}\}$. The discount factor is embodied in the utility function.

The consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ \sum_{k=1}^{T} \sum_{\upsilon_k=1}^{\overline{m}_k} \rho_k^{\upsilon_k} \delta_1^k u^{k(\upsilon_k)}(x_k) \right\}$$

$$= E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ u^{1(1)}(x_1) + \sum_{k=2}^{T} \sum_{\upsilon_k=1}^{\overline{m}_k} \rho_k^{\upsilon_k} \delta_1^k u^{k(\upsilon_k)}(x_k) \right\}$$

subject to the budget constraint characterized by the wealth dynamic

$$W_{k+1} = W_k - \sum_{h=1}^{n_k} p_k^h x_k^h + r(W_k - \sum_{h=1}^{n_k} p_k^h x_k^h) + Y_{k+1}, \qquad W_1 = W_1^0 .$$

C14. Dynamic Slutsky Equation under Stochastic Life-span and Preferences

$$\frac{\partial \varphi_\ell^{(\upsilon_\ell)h}(W_\ell^0, p)}{\partial p_\ell^{i_\ell}} = \frac{\partial \psi_\ell^{(\upsilon_\ell)h}(W_\ell^0, p)}{\partial p_\ell^{i_\ell}} - \frac{\partial \varphi_\ell^{(\upsilon_\ell)h}(W_\ell^0, p)}{\partial W_\ell^0} \varphi_\ell^{i_\ell}(W_\ell^0, p),$$

$$\frac{\partial \varphi_\ell^{(\upsilon_\ell)h}(W_\ell^0, p)}{\partial p_k^{i_k}} = \frac{\partial \psi_\ell^{(\upsilon_\ell)h}(W_\ell^0, p)}{\partial p_k^{i_k}} - \frac{\partial \varphi_\ell^{(\upsilon_\ell)h}(W_\ell^0, p)}{\partial W_\ell^0} \sum_{\upsilon_{\ell+1}=1}^{\overline{m}_{\ell+1}} \rho_{\ell+1}^{\upsilon_{\ell+1}} \sum_{\upsilon_{\ell+2}=1}^{\overline{m}_{\ell+2}} \rho_{\ell+2}^{\upsilon_{\ell+2}} \cdots$$

$$\cdots \sum_{\upsilon_k=1}^{\overline{m}_k} \rho_k^{\upsilon_k} \ \delta_{\ell+1}^k \ \frac{\partial v^{(\upsilon_k)k}(W_k^{\upsilon_\ell \upsilon_{\ell+1}\cdots\upsilon_{k-1}}, p)}{\partial W_k^{\upsilon_\ell \upsilon_{\ell+1}\cdots\upsilon_{k-1}}} \ \varphi_k^{i_k}(W_k^{\upsilon_\ell \upsilon_{\ell+1}\cdots\upsilon_{k-1}}, p)\ (1+r)^{-(\ell-k)}$$

$$\div \left[\sum_{\varpi_{\ell+1}=1}^{\overline{m}_{\ell+1}} \rho_{\ell+1}^{\varpi_{\ell+1}} \sum_{\varpi_{\ell+2}=1}^{\overline{m}_{\ell+2}} \rho_{\ell+2}^{\varpi_{\ell+2}} \cdots \sum_{\varpi_h=1}^{\overline{m}_h} \rho_h^{\varpi_h} \ \delta_{\ell+1}^h \ \frac{\partial v^{h(\varpi_h)}(W_h^{\varpi_\ell \varpi_{\ell+1}\cdots\varpi_{h-1}}, p)}{\partial W_h^{\varpi_\ell \varpi_{\ell+1}\cdots\varpi_{h-1}}} \right],$$

for $\ell \in \{1,2,\cdots,T\}$, $k \in \{\ell+1, \ell+2, \cdots, T\}$, $i_k \in \{1,2,\cdots,n_k\}$, $h, i_\ell \in \{1,2,\cdots,n_\ell\}$ and $\upsilon_\ell \in \{1,2,\cdots,\overline{m}_\ell\}$. ■

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*
D.W.K. Yeung: Random Horizon Stocahstic dynamic Slutsky Equation under Preference Uncertainty, Applied Mathematical Sciences, Vol. 8, 2014, pp.7311-7340.

Problem C14: ***Corresponding Problem of Theorem C14.***

The dynamic Slutsky equation under stochastic life-span and preferences is derived from the consumer problem in which the consumer's life-span involves $\hat{T}$ periods where $\hat{T}$ is a random variable with range $\{1,2,\cdots,T\}$ and corresponding probabilities $\{\gamma_1, \gamma_2, \cdots, \gamma_T\}$. Conditional upon the reaching of period τ, the probability of the consumer's life-span would last up to periods $\tau, \tau+1, \cdots, T$ becomes respectively:

$$\frac{\gamma_\tau}{\sum_{\zeta=\tau}^{T}\gamma_\zeta}, \frac{\gamma_{\tau+1}}{\sum_{\zeta=\tau}^{T}\gamma_\zeta}, \cdots, \frac{\gamma_T}{\sum_{\zeta=\tau}^{T}\gamma_\zeta}.$$

The preference or utility function of the consumer in period 1 is known to be $u^{1(1)}(x_1)$. His future preferences are not known with certainty. In particular, his utility function in period $k \in \{2,3,\cdots,T\}$ is known to be $u^{k(\upsilon_k)}(x_k)$ with probability $\rho_k^{\upsilon_k}$ for $\upsilon_k \in \{1,2,\cdots,\overline{m}_k\}$ if he survives in period k. We use $\tilde{\upsilon}_k$ to denote the random variable with range $\upsilon_k \in \{1,2,\cdots,\overline{m}_k\}$ and corresponding probabilities $\{\rho_k^1, \rho_k^2, \cdots, \rho_k^{\overline{m}_k}\}$. The discount factor is embodied in the utility function.

The consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ \sum_{\hat{T}=1}^{T}\gamma_{\hat{T}} \sum_{k=1}^{\hat{T}} \sum_{\upsilon_k=1}^{\overline{m}_k} \rho_k^{\upsilon_k}\delta_1^k u^{k(\upsilon_k)}(x_k) \right\}$$

$$= E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ u^{1(1)}(x_1) + \sum_{\hat{T}=2}^{T}\gamma_{\hat{T}} \sum_{k=2}^{\hat{T}} \sum_{\upsilon_k=1}^{\overline{m}_k} \rho_k^{\upsilon_k}\delta_1^k u^{k(\upsilon_k)}(x_k) \right\}$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1} = W_k - \sum_{h=1}^{n_k} p_k^h x_k^h + r(W_k - \sum_{h=1}^{n_k} p_k^h x_k^h) + Y_{k+1}, \qquad W_1 = W_1^0.$$

C15. Dynamic Slutsky Equation under Stochastic Income and Preferences

$$\frac{\partial\varphi_\ell^{(\upsilon_\ell)h}(W_\ell^0,p)}{\partial p_\ell^{i_\ell}} = \frac{\partial\psi_\ell^{(\upsilon_\ell)h}(W_\ell^0,p)}{\partial p_\ell^{i_\ell}} - \frac{\partial\varphi_\ell^{(\upsilon_\ell)h}(W_\ell^0,p)}{\partial W_\ell^0}\ \varphi_\ell^{i_\ell}(W_\ell^0,p),$$

$$\frac{\partial\varphi_\ell^{(\upsilon_\ell)h}(W_\ell^0,p)}{\partial p_k^{i_k}} = \frac{\partial\psi_\ell^{(\upsilon_\ell)h}(W_\ell^0,p)}{\partial p_k^{i_k}} - \frac{\partial\varphi_\ell^{(\upsilon_\ell)h}(W_\ell^0,p)}{\partial W_\ell^0}$$

$$\times \sum_{j_{\ell+1}=1}^{m_{\ell+1}}\lambda_{\ell+1}^{j_{\ell+1}} \sum_{j_{\ell+2}=1}^{m_{\ell+2}}\lambda_{\ell+2}^{j_{\ell+2}} \cdots \sum_{j_\ell=1}^{m_\ell}\lambda_k^{j_k} \sum_{\upsilon_{\ell+1}=1}^{\overline{m}_{\ell+1}}\rho_{\ell+1}^{\upsilon_{\ell+1}} \sum_{\upsilon_{\ell+2}=1}^{\overline{m}_{\ell+2}}\rho_{\ell+2}^{\upsilon_{\ell+2}} \cdots$$

$$\cdots \sum_{\upsilon_k=1}^{\overline{m}_k}\rho_k^{\upsilon_k}\ \delta_{\ell+1}^k\ \frac{\partial v^{(\upsilon_k)k}(W_k^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_k^{j_k};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{k-1}},p)}{\partial W_k^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_k^{j_k};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{k-1}}}\ \varphi_k^{i_k}(W_k^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_k^{j_k};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{k-1}},p)$$

$$(1+r)^{-(\ell-k)}$$

$$\div\Bigg[\sum_{w_{\ell+1}=1}^{m_{\ell+1}}\lambda_{\ell+1}^{w_{\ell+1}} \sum_{w_{\ell+2}=1}^{m_{\ell+2}}\lambda_{\ell+2}^{w_{\ell+2}} \cdots \sum_{w_k=1}^{m_k}\lambda_k^{w_k} \sum_{\varpi_{\ell+1}=1}^{\overline{m}_{\ell+1}}\rho_{\ell+1}^{\varpi_{\ell+1}} \sum_{\varpi_{\ell+2}=1}^{\overline{m}_{\ell+2}}\rho_{\ell+2}^{\varpi_{\ell+2}} \cdots$$

$$\cdots \sum_{\varpi_h=1}^{\overline{m}_h}\rho_h^{\varpi_h}\ \delta_{\ell+1}^h\ \frac{\partial v^{h(\varpi_h)}(W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_h^{w_h};\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}},p)}{\partial W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_h^{w_h};\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}}}\Bigg],$$

for $\ell \in \{1,2,\cdots,T\}$, $k \in \{\ell+1,\ell+2,\cdots,T\}$, $i_k \in \{1,2,\cdots,n_k\}$, $h, i_\ell \in \{1,2,\cdots,n_\ell\}$ and $\upsilon_\ell \in \{1,2,\cdots,\overline{m}_\ell\}$. ■

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*
D.W.K. Yeung: Random Horizon Stocahstic dynamic Slutsky Equation under

Problem C15: ***Corresponding Problem of Theorem C15.***
The dynamic Slutsky equation under stochastic income and preferences is derived from the consumer problem in which the preference or utility function of the consumer in period 1 is known to be $u^{1(1)}(x_1)$. His future preferences are not known with certainty. In particular, his utility function in period $k \in \{2,3,\cdots,T\}$ is known to be $u^{k(\upsilon_k)}(x_k)$ with probability $\rho_k^{\upsilon_k}$ for $\upsilon_k \in \{1,2,\cdots,\overline{m}_k\}$. We use $\widetilde{\upsilon}_k$ to denote the random variable with range $\upsilon_k \in \{1,2,\cdots,\overline{m}_k\}$ and corresponding probabilities $\{\rho_k^1,\rho_k^2,\cdots,\rho_k^{\overline{m}_k}\}$. The discount factor is embodied in the utility function.

The consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ \sum_{k=1}^{T} \sum_{\upsilon_k=1}^{\overline{m}_k} \rho_k^{\upsilon_k}\delta_1^k u^{k(\upsilon_k)}(x_k) \right\}$$

$$= E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ u^{1(1)}(x_1) + \sum_{k=2}^{T} \sum_{\upsilon_k=1}^{\overline{m}_k} \rho_k^{\upsilon_k}\delta_1^k u^{k(\upsilon_k)}(x_k) \right\}$$

subject to the budget constraint characterized by the wealth dynamic

$$W_{k+1} = W_k - \sum_{h=1}^{n_k} p_k^h x_k^h + r(W_k - \sum_{h=1}^{n_k} p_k^h x_k^h) + \theta_{k+1}, \qquad W_1 = W_1^0.$$

where

θ_k is the random income that the consumer will receive in period k; and θ_k, for $k \in \{2,\cdots,T\}$, is a set of statistically independent random variables, and $E_{\theta_1,\theta_2,\cdots,\theta_T}$ is the expectation operation with respect to the statistics of $\theta_2,\theta_3,\cdots,\theta_T$. The random variable θ_k has a non-negative range $\{\theta_k^1,\theta_k^2,\cdots,\theta_k^{m_k}\}$ with corresponding probabilities $\{\lambda_k^1,\lambda_k^2,\cdots,\lambda_k^{m_k}\}$, for $k \in \{2,\cdots,T\}$.

C16. Dynamic Slutsky Equation under Stochastic Income, Life-span and Preferences

$$\frac{\partial \varphi_\ell^{(\upsilon_\ell)h}(W_\ell^0,p)}{\partial p_\ell^{i_\ell}} = \frac{\partial \psi_\ell^{(\upsilon_\ell)h}(W_\ell^0,p)}{\partial p_\ell^{i_\ell}} - \frac{\partial \varphi_\ell^{(\upsilon_\ell)h}(W_\ell^0,p)}{\partial W_\ell^0}\ \varphi_\ell^{i_\ell}(W_\ell^0,p),$$

$$\frac{\partial \varphi_\ell^{(\upsilon_\ell)h}(W_\ell^0,p)}{\partial p_k^{i_k}} = \frac{\partial \psi_\ell^{(\upsilon_\ell)h}(W_\ell^0,p)}{\partial p_k^{i_k}} - \frac{\partial \varphi_\ell^{(\upsilon_\ell)h}(W_\ell^0,p)}{\partial W_\ell^0}$$

$$\times \sum_{j_{\ell+1}=1}^{m_{\ell+1}} \lambda_{\ell+1}^{j_{\ell+1}} \sum_{j_{\ell+2}=1}^{m_{\ell+2}} \lambda_{\ell+2}^{j_{\ell+2}} \cdots \sum_{j_\ell=1}^{m_\ell} \lambda_k^{j_k} \sum_{\upsilon_{\ell+1}=1}^{\overline{m}_{\ell+1}} \rho_{\ell+1}^{\upsilon_{\ell+1}} \sum_{\upsilon_{\ell+2}=1}^{\overline{m}_{\ell+2}} \rho_{\ell+2}^{\upsilon_{\ell+2}} \cdots$$

$$\cdots \sum_{\upsilon_k=1}^{\overline{m}_k} \rho_k^{\upsilon_k} \quad \delta_{\ell+1}^k \quad \frac{\partial v^{(\upsilon_k)k}(W_k^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_k^{jk};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{k-1}},p)}{\partial W_k^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_k^{jk};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{k-1}}} \quad \varphi_k^{i_k}(W_k^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_k^{jk};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{k-1}},p)$$

$$(1+r)^{-(\ell-k)}$$

$$\div \Bigg[\sum_{w_{\ell+1}=1}^{m_{\ell+1}} \lambda_{\ell+1}^{w_{\ell+1}} \sum_{w_{\ell+2}=1}^{m_{\ell+2}} \lambda_{\ell+2}^{w_{\ell+2}} \cdots \sum_{w_k=1}^{m_k} \lambda_k^{w_k} \sum_{\varpi_{\ell+1}=1}^{\overline{m}_{\ell+1}} \rho_{\ell+1}^{\varpi_{\ell+1}} \sum_{\varpi_{\ell+2}=1}^{\overline{m}_{\ell+2}} \rho_{\ell+2}^{\varpi_{\ell+2}} \cdots$$

$$\cdots \sum_{\varpi_h=1}^{\overline{m}_h} \rho_h^{\varpi_h} \ \delta_{\ell+1}^h \frac{\partial v^{h(\varpi_h)}(W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_h^{w_h};\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}}, p)}{\partial W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_h^{w_h};\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}}} \Bigg],$$

for $\ell \in \{1,2,\cdots,T\}$, $k \in \{\ell+1, \ell+2,\cdots,T\}$, $i_k \in \{1,2,\cdots,n_k\}$, $h, i_\ell \in \{1,2,\cdots,n_\ell\}$ and $\upsilon_\ell \in \{1,2,\cdots,\overline{m}_\ell\}$. ■

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*
D.W.K. Yeung: Random Horizon Stocahstic dynamic Slutsky Equation under Preference Uncertainty, Applied Mathematical Sciences, Vol. 8, 2014, pp.7311-7340.

Problem C16: ***Corresponding Problem of Theorem C16.***

The dynamic Slutsky equation under stochastic income, life-span and preferences is derived from the consumer problem in which the consumer's life-span involves $\hat{T}$ periods where $\hat{T}$ is a random variable with range $\{1,2,\cdots,T\}$ and corresponding probabilities $\{\gamma_1, \gamma_2, \cdots, \gamma_T\}$. Conditional upon the reaching of period τ, the probability of the consumer's life-span would last up to periods $\tau, \tau+1, \cdots, T$ becomes respectively:

$$\frac{\gamma_\tau}{\sum_{\zeta=\tau}^{T} \gamma_\zeta}, \frac{\gamma_{\tau+1}}{\sum_{\zeta=\tau}^{T} \gamma_\zeta}, \cdots, \frac{\gamma_T}{\sum_{\zeta=\tau}^{T} \gamma_\zeta}.$$

The preference or utility function of the consumer in period 1 is known to be $u^{1(1)}(x_1)$. His future preferences are not known with certainty. In particular, his utility function in period $k \in \{2,3,\cdots,T\}$ is known to be $u^{k(\upsilon_k)}(x_k)$ with probability $\rho_k^{\upsilon_k}$ for $\upsilon_k \in \{1,2,\cdots,\overline{m}_k\}$ if he survives in period k. We use $\tilde{\upsilon}_k$ to denote the random variable with range $\upsilon_k \in \{1,2,\cdots,\overline{m}_k\}$ and corresponding probabilities $\{\rho_k^1, \rho_k^2, \cdots, \rho_k^{\overline{m}_k}\}$. The discount factor is embodied in the utility function.

The consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ \sum_{\hat{T}=1}^{T} \gamma_{\hat{T}} \sum_{k=1}^{\hat{T}} \sum_{\upsilon_k=1}^{\overline{m}_k} \rho_k^{\upsilon_k} \delta_1^k u^{k(\upsilon_k)}(x_k) \right\}$$

$$= E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ u^{1(1)}(x_1) + \sum_{\hat{T}=2}^{T} \gamma_{\hat{T}} \sum_{k=2}^{\hat{T}} \sum_{\upsilon_k=1}^{\overline{m}_k} \rho_k^{\upsilon_k} \delta_1^k u^{k(\upsilon_k)}(x_k) \right\},$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1} = W_k - \sum_{h=1}^{n_k} p_k^h x_k^h + r(W_k - \sum_{h=1}^{n_k} p_k^h x_k^h) + \theta_{k+1}, \qquad W_1 = W_1^0.$$

where

θ_k is the random income that the consumer will receive in period k; and θ_k, for $k \in \{2,\cdots,T\}$, is a set of statistically independent random variables, and $E_{\theta_1,\theta_2,\cdots,\theta_T}$ is the expectation operation with respect to the statistics of $\theta_2, \theta_3, \cdots, \theta_T$. The random variable θ_k has a non-negative range $\{\theta_k^1, \theta_k^2, \cdots, \theta_k^{m_k}\}$ with corresponding probabilities $\{\lambda_k^1, \lambda_k^2, \cdots, \lambda_k^{m_k}\}$, for $k \in \{2,\cdots,T\}$.

D. Biological Population Density Functions

D1. Stationary Density Function of Generalized Stochastic Food-chain of the Lotka-Volterra-Yeung Type

The function

$$\psi(N) = m\prod_{i=1}^{n}\frac{1}{N_i}\exp\left[\left[2A_i \ln N_i - 2F_i(\ln N_i) + 2F_i(0)\right]\big/\sigma^2\right]$$

gives the stationary probability density of species $N_1, N_2, \ldots, N_n$ of the generalized Lotka-Volterra-Yeung type of stochastic food-chain:

$$dN_1(t) = [\alpha_1 N_1(t) - b_1 N_1(t) f_1(N_1(t)) - v_1 N_1(t) f_2(N_2(t))]\, dt + \sigma\sqrt{b_1} N_1(t)dz(t),$$

$$dN_2(t) = [\alpha_2 N_2(t) - b_2 N_2(t) f_2(N_2(t)) - v_2 N_2(t) f_3(N_3(t))$$
$$+ v_1 N_2(t) f_1(N_1(t))]dt + \sigma\sqrt{b_2} N_2(t)dz(t),$$

$$dN_3(t) = [\alpha_3 N_3(t) - b_3 N_3(t) f_3(N_3(t)) - v_3 N_3(t) f_4(N_4(t))$$
$$+ v_2 N_3(t) f_2(N_2(t))]dt + \sigma\sqrt{b_3} N_3(t)dz(t),$$

$$\vdots \qquad \vdots$$

$$dN_{n-1}(t) = [\alpha_{n-1} N_{n-1}(t) - b_{n-1} N_{n-1}(t) f_{n-1}(N_{n-1}(t)) - v_{n-1} N_{n-1}(t) f_n(N_n(t))$$
$$+ v_{n-2} N_{n-1}(t) f_{n-2}(N_{n-2}(t))]\, dt + \sigma\sqrt{b_{n-1}} N_{n-1}(t)dz(t),$$

$$dN_n(t) = [\alpha_n N_n(t) - b_n N_n(t) f_n(N_n(t)) + v_{n-1} N_n(t) f_{n-1}(N_{n-1}(t))]dt$$
$$+ \sigma\sqrt{b_n} N_n(t)dz(t),$$

where $N_i(t)$ is the population level of the species in the i^{th} trophic level at time t;

v_i for $i \in [1,2,\cdots,n-1]$ are positive constants, b_1 is positive and b_i for $i \in [2,3,\cdots,n]$ are nonnegative constants;

$\alpha_1 > 0$, and α_i for $i \in [2,3,\cdots,n]$ are constants with α_i being positive when $b_i > 0$ and negative when $b_i = 0$;

$f_i(0) = 0$ and $f_i(N_i) > 0$ for positive values of N_i, and $f_i(N_i)$ is a continuous differentiable and monotonically increasing in N_i, and $f_i(e^s)$ is an integrable function yielding $\int_0^{x_i} f_i(e^s)ds = F_i(x_i) - F_i(0)$, for $i = 1, 2, \ldots, n$;

and

$A_1, A_2, \cdots, A_n$ satisfies

$b_1 A_1 + v_1 A_2 = \omega_1,$

$-v_1 A_1 + b_2 A_2 + v_2 A_3 = \omega_2,$

$-v_2 A_2 + b_3 A_3 + v_3 A_4 = \omega_3,$

$\vdots \qquad \vdots$

$-v_{n-2} A_{n-2} + b_{n-1} A_{n-1} + v_{n-1} A_n = \omega_{n-1},$

$-v_{n-1} A_{n-1} + b_n A_n = \omega_n.$

References: *D.W.K. Yeung: An Explicit Density Function for a Generalized*

Stochastic Food-chain of the Lotka-Volterra-Yeung Type, Stochastic Analysis and Applications, Vol. 27, 2009, pp.16-23.

Note:

Using the generalized density function D1 one can also obtain the stationary density function of the stochastic Lotka-Volterra food-chain in Yeung (1988):

$$dN_1(t) = [a_1 N_1(t) - bN_1^2(t) - c_1 N_1(t)(N_2(t))]\,dt + \varepsilon N_1(t)dz(t),$$
$$dN_i(t) = [-a_i N_i(t) - c_i N_i(t)(N_{i+1}(t)) + \gamma_i N_i(t) N_{i-1}(t)]dt,$$
$$\text{for } i = 2,3,\cdots,n-1,$$
$$dN_n(t) = [-a_n N_n(t) + \gamma_n N_n(t) N_{n-1}(t)]dt,$$

where $N_i(t)$ is the population level of the species in the i^{th} trophic level at time t, $z(t)$ is a standard Wiener process, with $E(dz_i) = 0$, $E(dz_i^2) = dt$ and $E(dtdz) = 0$, b, a_i for $i \in [1,2,3,\cdots,n]$ and c_i for $i \in [1,2,3,\cdots,n-1]$ and γ_i for $i \in [2,3,\cdots,n]$ are positive constants, and ε is a constant.

Similarly, using the generalized density function D1 one can also obtain the stationary density function of the prey species N_1 and the predator species N_2 of the predator prey system in Yeung (1986):

$$dN_1(t) = [a_1 N_1(t) - bN_1^2(t) - c_1 N_1(t)(N_2(t))]\,dt + \varepsilon N_1(t)dz(t),$$
$$dN_2(t) = [-a_2 N_2(t) + \gamma N_2(t) N_1(t)]dt,$$

where a_1, a_2, b, γ and ε are positive constants, and $z(t)$ is a standard Wiener process, with $E(dz_i) = 0$, $E(dz_i^2) = dt$.

References: *D.W.K. Yeung: Exact Solutions for Steady-State Probability Distribution of a Simple Stochastic Lotka Volterra Food Chain. Stochastic Analysis and Applications, Vol. VI, 1988, pp. 103-116.*

D.W.K. Yeung: Optimal Management of Replenishable Resources in a Predator-Prey System with Randomly Fluctuating Population. Mathematical Biosciences, Vol. 78, 1986, pp. 91-105.

E. Number Theory

E1. The Number of Embedded Coalitions

The number of embedded coalitions in a n-person game is:

$$\mathrm{Y}(1) = \sum_{t=0}^{0} \binom{1}{t} = \binom{1}{0} = 1, \qquad \text{for } n = 1;$$

$$\mathrm{Y}(2) = \sum_{t=0}^{1} \binom{2}{t} = \binom{2}{1} + \binom{2}{0} = 3, \qquad \text{for } n = 2;$$

$$\Upsilon(n) = \sum_{t=2}^{n-1} \binom{n}{t} \left(\sum_{k=1}^{t-1} \Upsilon(k) \right) + \sum_{t=0}^{n-1} \binom{n}{t}, \quad \text{for } n \geq 3. \qquad \blacksquare$$

Problem E1: ***Corresponding Problem of Theorem E1.***

Let $N = \{1,2,\cdots,n\}$ be a finite set of n players in a $n-$person game. The subsets of N are coalitions. A partition Λ is formed by disjoint non-empty subsets of N representing a way that these n players are joined. Given a partition Λ and a coalition $S \subset N$, the pair (S, Λ) is called an embedded coalition, that is the coalition S embedded in partition Λ. The Bell (1934) number, denoted by $\beta(n)$, gives the number of partitions in a $n-$person game. The number of embedded coalitions in a partition is the number of subsets formed in that partition. The total number of embedded coalitions $Y(n)$ in a $n-$ person game is the sum of the numbers of embedded coalitions in the $\beta(n)$ partitions of N.

References: *D.W.K. Yeung: Recursive Sequences Identifying the Number of Embedded Coalitions, International Game Theory Review, Vol. 10(1), 2008, pp.129-136.*
E. T. Bell [1934] Exponential numbers, American Mathematical Monthly 41, 411-419, 1934.

E2. The Number of Embedded Coalitions where the position of the individual player counts

The number of embedded coalitions in a n-person game where the position of the individual player counts is:

$$\varphi(1) = 1\sum_{t=0}^{0} \binom{1}{t} = 1, \qquad \text{for } n = 1;$$

$$\varphi(2) = 2\sum_{t=0}^{1} \binom{2}{t} = 6, \qquad \text{for } n = 2;$$

$$\varphi(n) = n! \left[\sum_{t=2}^{n-1} \binom{n}{t} \left(\sum_{k=1}^{t-1} \frac{\varphi(k)}{k!} \right) + \sum_{t=0}^{n-1} \binom{n}{t} \right], \text{ for } n \geq 3. \qquad \blacksquare$$

References: *D.W.K. Yeung, E.L.H. Ku and P.M. Yeung: A Recursive Sequence for the Number of Positioned Partitions, International Journal of Algebra, Vol. 2, 2008, pp.181-185.*

Problem E2: ***Corresponding Problem of Theorem E2.***

Consider the problem in Problem E1 in which the position of the individual player in a embedded coalition counts. The total number of embedded coalitions $\varphi(n)$ in a is the sum of the numbers of embedded coalitions with the positions of individual players count in the $\beta(n)$ partitions of N.